A REFINED BLOCH GROUP AND THE THIRD HOMOLOGY OF SL_2 OF A FIELD

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Abstract. We use the properties of the refined Bloch group to prove that H_3 of SL_2 of a global field is never finitely-generated, and to calculate H_3 of SL_2 of local fields with odd residue characteristic up to some 2-torsion. We also give lower bounds for the 3-rank of the homology groups $\mathrm{H}_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})$.

Contents

1. Introduction

In [10], Chih-Han Sah quotes S. Lichtenbaum who mentions our lack of knowledge of the precise structure of $\mathrm{H}_3(\mathrm{SL}_2(\mathbb{Q}), \mathbb{Z})$ as an example of the unsatisfactory state of our understanding of the homology of linear groups. This was nearly twenty five years ago, and to the author's knowledge the precise structure of this group is still unknown. We will return to this question below. (Observe, however, that $\mathrm{H}_3(\mathrm{SL}_n(\mathbb{Q}), \mathbb{Z}) \cong K_3^{\mathrm{ind}}(\mathbb{Q}) \cong \mathbb{Z}/24$ for all $n \geq 3$ - by the results of [5], for example.)

In this article, we study the structure of the third homology of SL_2 of fields by using the properties of the *refined Bloch group* of the field, which was introduced in [4]. We are particularly interested in understanding $\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z})$ as a *functor* of F, and its possible relation to other functors in K-theory and algebraic geometry.

What are now referred to as *Bloch groups* of fields first appeared in the work of S. Bloch in the late 1970s (see [1] for the lecture notes) as a way of constructing explicit maps - and, in particular, regulators - on K_3 of fields. In the 1980s, they were studied by Dupont and Sah (under the name *scissors congruence group*) because of their connection with 3-dimensional hyperbolic geometry ([2], [10]). This connection is still actively studied today: Bloch groups of number fields are targets for *Bloch invariants* of certain finite-volume oriented hyperbolic

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3-manifolds ([8], [9], [3]). These invariants are amenable to explicit calculation and are related in a known way to the Chern-Simons invariant. There are also intriguing connections between Bloch groups, conformal field theories and even modular form theory ([12],[7]).

The precise relationship between the Bloch group and K-theory was established via their mutual connection to the homology of linear groups. These connections were greatly clarified and exploited in the work of Suslin ([11]): For an infinite field F, the Bloch group, $\mathcal{B}(F)$, arises naturally as a quotient of $\mathrm{H}_3(\mathrm{GL}_2(F),\mathbb{Z})$. Of course, $K_3(F)$ admits a Hurewicz homomorphism to $\mathrm{H}_3(\mathrm{GL}(F),\mathbb{Z}) = \mathrm{H}_3(\mathrm{GL}_3(F),\mathbb{Z})$. Suslin proved that there is a natural short exact sequence

$$0 \to \widetilde{\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F,\mu_F)} \to K_3^{\mathrm{ind}}(F) \to \mathcal{B}(F) \to 0$$

where $\widetilde{\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F,\mu_F)}$ is the unique nontrivial extension of $\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F,\mu_F)$ by $\mathbb{Z}/2$ and

$$K_3^{\mathrm{ind}}(F) = \mathrm{Coker}(K_3^M(F) \to K_3(F)).$$

These results of Suslin were extended to finite fields (with at least 4 elements) in [4].

In a letter to Sah, Suslin asked the question whether the composite $\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}) \to \mathrm{H}_3(\mathrm{GL}_2(F),\mathbb{Z}) \to K_3^{\mathrm{ind}}(F)$ induces an isomorphism

$$\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z})_{F^\times} \cong K_3^{\mathrm{ind}}(F).$$

The current state of knowledge on this question is that the map is surjective ([5]) and that the induced map

$$\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}])_{F^\times} \to K_3^{\mathrm{ind}}(F)[\tfrac{1}{2}]$$

is an isomorphism, where $A[\frac{1}{2}]$ denotes $A \otimes \mathbb{Z}[\frac{1}{2}]$ for any abelian group A ([6]).

The homology groups of the special linear groups $\mathrm{SL}_n(F)$ are naturally modules over the group ring $\mathbb{Z}[F^\times]$ via the short exact sequences

$$1 \longrightarrow \mathrm{SL}_n(F) \longrightarrow \mathrm{GL}_n(F) \xrightarrow{\det} F^\times \longrightarrow 1.$$

Since the scalar matrices $a \cdot I_n$ are central and have determinant a^n, it, of course, follows that $(F^\times)^n$ acts trivially on $\mathrm{H}_k(\mathrm{SL}_n(F),\mathbb{Z})$. In particular, the groups $\mathrm{H}_k(\mathrm{SL}_2(F),\mathbb{Z})$ are modules for the group ring $\mathrm{R}_F := \mathbb{Z}[F^\times/(F^\times)^2]$.

When $n > k$ (or $n \geq k$ when k is odd), we are in the range of stability (see [10], [5] and [**?**]) and this module structure is necessarily trivial. But below the range of stability, the module structure appears to be nontrivial and interesting. For example, the unstable groups $\mathrm{H}_{2n}(\mathrm{SL}_{2n}(F),\mathbb{Z})$ are modules over the Grothendieck-Witt ring of the field (which is a quotient of R_F) and surject onto the Milnor-Witt K-theory groups $K_{2n}^{\mathrm{MW}}(F)$ ([**?**]).

In [4] the author defined a *refined Bloch group*, $\mathcal{RB}(F)$, of a field F which was shown to have the following properties:

(1) The group $\mathcal{RB}(F)$ is an R_F-module and there is a natural surjective homomorphism of R_F-modules

$$\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}) \twoheadrightarrow \mathcal{RB}(F)$$

(2) This induces a commutative diagram (of $\mathrm{R}_F[\frac{1}{2}]$-modules) with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}(\mu_F,\mu_F)[\tfrac{1}{2}] & \longrightarrow & \mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}]) & \longrightarrow & \mathcal{RB}(F)[\tfrac{1}{2}] & \longrightarrow & 0\\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}(\mu_F,\mu_F)[\tfrac{1}{2}] & \longrightarrow & K_3^{\mathrm{ind}}(F)[\tfrac{1}{2}] & \longrightarrow & \mathcal{B}(F)[\tfrac{1}{2}] & \longrightarrow & 0
\end{array}$$

Here R_F acts trivially on the bottom row and furthermore

(3) On taking $F^\times$-coinvariants, the top row becomes isomorphic to the bottom row. In particular, $\mathcal{RB}(F)[\frac{1}{2}]_{F^\times} \cong \mathcal{B}(F)[\frac{1}{2}]$ and

$$\mathcal{I}_F\mathcal{RB}(F)[\tfrac{1}{2}] \cong \mathcal{I}_F\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}]) = \mathrm{Ker}(\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}]) \to K_3^{\mathrm{ind}}(F)[\tfrac{1}{2}])$$

where $\mathcal{I}_F$ denotes the augmentation ideal of the group ring R_F.

(The group $\mathcal{I}_F\mathcal{RB}(F)[\frac{1}{2}]$ is also the kernel of the stabilization homomorphism

$$\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}]) \to \mathrm{H}_3(\mathrm{SL}_3(F),\mathbb{Z}[\tfrac{1}{2}]).$$

The cokernel of this map is the third Milnor K-group $K_3^{\mathrm{M}}(F)[\frac{1}{2}]$ and the image is isomorphic to $K_3^{\mathrm{ind}}(F)[\frac{1}{2}]$.)

The main result of the current article (Theorem 4.4 and its corollaries) tells us that given a valuation v on the field F with residue field k, there are surjective reduction or specialization homomorphisms

$$\mathcal{RB}(F) \twoheadrightarrow \widehat{\mathcal{RP}(k)}$$

where $\widehat{\mathcal{RP}(k)}$ is a certain quotient of the refined *pre-Bloch group*, $\mathcal{RP}(k)$, of the field k.

In particular, if $a \in F^\times$ and $v(a)$ is not a multiple of 2, there is a specialization homomorphism which induces a surjection

$$\mathrm{e}_a^-\mathcal{RB}(F)[\tfrac{1}{2}] \twoheadrightarrow \widehat{\mathcal{P}(k)}[\tfrac{1}{2}]$$

where

$$\mathrm{e}_a^- = \frac{1-\langle a\rangle}{2} \in \mathcal{I}_F[\tfrac{1}{2}]$$

and $\langle a\rangle$ denotes the square class of a in R_F and $\widehat{\mathcal{P}(k)}$ is a certain quotient of the classical pre-Bloch group, $\mathcal{P}(k)$, of the residue field.

Using these results, we can prove that $\mathcal{I}_F\mathcal{RB}(F)[\frac{1}{2}]$ is large if F is a field with many valuations. In particular, it follows that $\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z})$ is not finitely generated for any global field F. (By contrast, if F is a global field then $\mathrm{H}_3(\mathrm{SL}_3(F),\mathbb{Z}) = K_3^{\mathrm{ind}}(F)$ is well-known to be finitely generated.)

For example, (see Theorem 5.1) if F is a global field whose class group has odd order then there is a natural surjection

$$\mathcal{I}_F\mathcal{RB}(F)[\tfrac{1}{2}] \twoheadrightarrow \bigoplus_v \mathcal{B}(k_v)[\tfrac{1}{2}]$$

where v runs through all the finite places of F and k_v is the residue field at v. (By the results of [4], if $\mathbb{F}_q$ is the finite field with q elements, then $\mathcal{B}(\mathbb{F}_q)$ is cyclic of order $(q+1)/2$ or $q+1$ according as q is odd or even.)

As another application, we also use the techniques developed to construct explicitly non-trivial $\mathbb{F}_3$-vectorspaces of known dimension inside groups of the form $\mathrm{H}_3(\mathrm{SL}_2(O_S),\mathbb{Z})$ where O_S is a ring of S-integers, and thus to give lower bounds on the 3-ranks of such groups. We hope that these techniques will be useful in proving more general results of this type in the future.

Finally, we use these specialization maps, together with the basic algebraic properties of the refined Bloch groups, developed in section 3 below, to give a calculation of $\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\frac{1}{2}])$ for local fields F with finite residue field of odd order (Theorem 6.14). In particular, for such fields we have

$$\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}]) \cong K_3^{\mathrm{ind}}(F)[\tfrac{1}{2}] \oplus \mathcal{B}(k)[\tfrac{1}{2}]$$

where k is the residue field. Here the right-hand side is an R_F module with $F^\times$ acting trivially on the first factor while any uniformizer acts as -1 on the second factor.

To return to our opening remarks, it follows from the results here that there is a natural surjection

$$\mathrm{H}_3(\mathrm{SL}_2(\mathbb{Q}),\mathbb{Z})\otimes\mathbb{Z}[\tfrac{1}{2}] = \mathrm{H}_3(\mathrm{SL}_2(\mathbb{Q}),\mathbb{Z}[\tfrac{1}{2}]) \twoheadrightarrow K_3^{\mathrm{ind}}(\mathbb{Q})[\tfrac{1}{2}]\oplus\left(\bigoplus_p \mathcal{B}(\mathbb{F}_p)[\tfrac{1}{2}]\right).$$

It is natural to ask whether this is an isomorphism and, furthermore, what adjustments, if any, need to be made to obtain a corresponding statement with integral coefficients.

2. Review of Bloch Groups

2.1. **Preliminaries and Notation.** For a field F, we let G_F denote the multiplicative group, $F^\times/(F^\times)^2$, of nonzero square classes of the field. For $x\in F^\times$, we will let $\langle x\rangle\in G_F$ denote the corresponding square class. Let R_F denote the integral group ring $\mathbb{Z}[G_F]$ of the group G_F. We will use the notation $\langle\langle x\rangle\rangle$ for the basis elements, $\langle x\rangle - 1$, of the augmentation ideal $\mathcal{I}_F$ of R_F.

For any $a\in F^\times$, we will let p_a^+ and p_a^- denote respectively the elements $1+\langle a\rangle$ and $1-\langle a\rangle$ in R_F.

For any abelian group A we will let $A[\tfrac{1}{2}]$ denote $A\otimes\mathbb{Z}[\tfrac{1}{2}]$. For an integer n, we will let n' denote the odd part of n. Thus if A is a finite abelian group of order n, then $A[\tfrac{1}{2}]$ is a finite abelian group of order n'.

We let e_a^+ and e_a^- denote respectively the mutually orthogonal idempotents

$$\mathrm{e}_a^+ := \frac{\mathrm{p}_a^+}{2} = \frac{1+\langle a\rangle}{2},\quad \mathrm{e}_a^- := \frac{\mathrm{p}_a^-}{2} = \frac{1-\langle a\rangle}{2}\in\mathrm{R}_F[\tfrac{1}{2}].$$

(Of course, these operators depend only on the class of a in G_F.)

2.2. **The classical Bloch group.** For a field F, with at least 4 elements, the *pre-Bloch group*, $\mathcal{P}(F)$, is the group generated by the elements $[x]$, $x\in F^\times\setminus\{1\}$, subject to the relations

$$R_{x,y}:\quad [x]-[y]+[y/x]-\left[(1-x^{-1})/(1-y^{-1})\right]+[(1-x)/(1-y)]\quad x\neq y.$$

Let $\mathrm{S}^2_{\mathbb{Z}}(F^\times)$ denote the group

$$\frac{F^\times\otimes_{\mathbb{Z}}F^\times}{< x\otimes y + y\otimes x | x,y\in F^\times >}$$

and denote by $x\circ y$ the image of $x\otimes y$ in $\mathrm{S}^2_{\mathbb{Z}}(F^\times)$.

The map

$$\lambda:\mathcal{P}(F)\to\mathrm{S}^2_{\mathbb{Z}}(F^\times),\quad [x]\mapsto(1-x)\circ x$$

is well-defined, and the *Bloch group of* F, $\mathcal{B}(F)\subset\mathcal{P}(F)$, is defined to be the kernel of λ.

2.3. **The refined Bloch group.** The *refined pre-Bloch group*, $\mathcal{RP}(F)$, of a field F which has at least 4 elements, is the R_F-module with generators $[x]$, $x\in F^\times$ subject to the relations $[1]=0$ and

$$S_{x,y}:\quad 0=[x]-[y]+\langle x\rangle[y/x]-\left\langle x^{-1}-1\right\rangle\left[(1-x^{-1})/(1-y^{-1})\right]+\langle 1-x\rangle[(1-x)/(1-y)],\quad x,y\neq 1$$

Of course, from the definition it follows immediately that $\mathcal{P}(F)=(\mathcal{RP}(F))_{F^\times}=\mathrm{H}_0(F^\times,\mathcal{RP}(F))$.

For any field F we define the R_F-module

$$\mathrm{RS}^2_{\mathbb{Z}}(F^\times):=\ \mathcal{I}_F^2\times_{\mathrm{Sym}^2_{\mathbb{F}_2}(G_F)}\mathrm{S}^2_{\mathbb{Z}}(F^\times)\subset\mathcal{I}_F^2\oplus\mathrm{S}^2_{\mathbb{Z}}(F^\times)$$

where $\mathrm{S}^2_{\mathbb{Z}}(F^\times)$ has the trivial R_F-module structure.

As an R_F-module, $\mathrm{RS}^2_{\mathbb{Z}}(F^\times)$ is generated by the elements

$$[a,b] := (\langle\!\langle a\rangle\!\rangle\langle\!\langle b\rangle\!\rangle, a\circ b) \in \mathrm{RS}^2_{\mathbb{Z}}(F^\times).$$

The *refined Bloch-Wigner homomorphism* Λ to be the R_F-module homomorphism

$$\Lambda : \mathcal{RP}(F) \to \mathrm{RS}^2_{\mathbb{Z}}(F), \qquad [x] \mapsto [1-x, x]$$

(which can be shown to be well-defined).

In view of the definition of $\mathrm{RS}^2_{\mathbb{Z}}(F^\times)$, we can express $\Lambda = (\lambda_1, \lambda_2)$ where $\lambda_1 : \mathcal{RP}(F) \to \mathcal{I}^2_F$ is the map $[x] \mapsto \langle\!\langle 1-x\rangle\!\rangle\langle\!\langle x\rangle\!\rangle$, and λ_2 is the composite

$$\mathcal{RP}(F) \twoheadrightarrow \mathcal{P}(F) \xrightarrow{\lambda} \mathrm{S}^2_{\mathbb{Z}}(F^\times).$$

Finally, we can define the *refined Bloch group* of the field F (with at least 4 elements) to be the R_F-module

$$\mathcal{RB}(F) := \mathrm{Ker}(\Lambda : \mathcal{RP}(F) \to \mathrm{RS}^2_{\mathbb{Z}}(F^\times)).$$

2.4. **The fields $\mathbb{F}_2$ and $\mathbb{F}_3$.** Throughout this paper it will be convenient for us to have (refined and classical) pre-Bloch and Bloch groups for the fields with 2 and 3 elements. For this reason, we introduce the following somewhat *ad hoc* definitions.

$\mathcal{P}(\mathbb{F}_2) = \mathcal{RP}(\mathbb{F}_2) = \mathcal{RB}(\mathbb{F}_2) = \mathcal{B}(\mathbb{F}_2)$ is simply an additive group of order 3 with distinguished generator, denoted $C_{\mathbb{F}_2}$.

$\mathcal{RP}(\mathbb{F}_3)$ is the cyclic $\mathrm{R}_{\mathbb{F}_3}$-module generated by the symbol $[-1]$ and subject to the one relation

$$0 = 2\cdot([-1] + \langle -1\rangle[-1]).$$

The homomorphism

$$\Lambda : \mathcal{RP}(\mathbb{F}_3) \to \mathrm{RS}^2_{\mathbb{Z}}(\mathbb{F}_3^\times) = \mathcal{I}^2_{\mathbb{F}_3} = 2\cdot\mathbb{Z}\langle\!\langle -1\rangle\!\rangle$$

is the $\mathrm{R}_{\mathbb{F}_3}$-homomorphism sending $[-1]$ to $\langle\!\langle -1\rangle\!\rangle^2 = -2\langle\!\langle -1\rangle\!\rangle$.

Then $\mathcal{RB}(\mathbb{F}_3) = \mathrm{Ker}(\Lambda)$ is the submodule of order 2 generated by $[-1] + \langle -1\rangle[-1]$.

Furthermore, we let $\mathcal{P}(\mathbb{F}_3) = \mathcal{RP}(\mathbb{F}_3)_{\mathbb{F}_3^\times}$. This is a cyclic $\mathbb{Z}$-module of order 4 with generator $[-1]$. Let $\lambda : \mathcal{P}(\mathbb{F}_3) \to \mathcal{I}^2_{\mathbb{F}_3}$ be the map $[-1] \mapsto -2\langle\!\langle -1\rangle\!\rangle$. Then $\mathcal{B}(\mathbb{F}_3) := \mathrm{Ker}(\lambda) = \mathcal{RB}(\mathbb{F}_3)$.

2.5. **The refined Bloch Group and $\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z})$.** We recall some results from [4]: The main result there is

Theorem 2.1. *Let F be a field with at least 4 elements.*

If F is infinite, there is a natural complex

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F) \to \mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}) \to \mathcal{RB}(F) \to 0.$$

which is exact everywhere except possibly at the middle term. The middle homology is annihilated by 4.

In particular, for any infinite field there is a natural short exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F)[\tfrac{1}{2}] \to \mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\tfrac{1}{2}]) \to \mathcal{RB}(F)[\tfrac{1}{2}] \to 0.$$

The following result is Corollary 5.1 in [4]:

Lemma 2.2. *Let F be an infinite field. Then the natural map $\mathcal{RB}(F) \to \mathcal{B}(F)$ is surjective and the induced map $\mathcal{RB}(F)_{F^\times} \to \mathcal{B}(F)$ has a 2-primary torsion kernel.*

Now for any field F, let

$$\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z})_0 := \mathrm{Ker}(\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}) \to K_3^{\mathrm{ind}}(F))$$

and

$$\mathcal{RB}(F)_0 := \mathrm{Ker}(\mathcal{RB}(F) \to \mathcal{B}(F))$$

The following is Lemma 5.2 in [4].

Lemma 2.3. *Let F be an infinite field. Then*

(1) $\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\frac{1}{2}])_0 = \mathcal{RB}(F)[\frac{1}{2}]_0$
(2) $\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\frac{1}{2}])_0 = \mathcal{I}_F\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\frac{1}{2}])$ *and* $\mathcal{RB}(F)[\frac{1}{2}]_0 = \mathcal{I}_F\mathcal{RB}(F)[\frac{1}{2}]$.
(3)

$$\begin{aligned}\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}])_0 &= \mathrm{Ker}(\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}]) \to \mathrm{H}_3(\mathrm{SL}_3(F),\mathbb{Z}[\tfrac{1}{2}]))\\ &= \mathrm{Ker}(\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}]) \to \mathrm{H}_3(\mathrm{GL}_2(F),\mathbb{Z}[\tfrac{1}{2}]))\end{aligned}$$

On the other hand, the corresponding results for finite fields are as follows (the results in [4] apply to fields with at least 4 elements, but it is straightforward to verify that they extend to the fields $\mathbb{F}_2$ and $\mathbb{F}_3$ with the definitions supplied above):

Lemma 2.4. *For a finite field k the natural map $\mathcal{RP}(k) \to \mathcal{P}(k)$ induces an isomorphism $\mathcal{RB}(k) \cong \mathcal{B}(k)$.*

(This is Lemma 7.1 in [4].)

For a field F, we let $\mathrm{Tor}_1^{\mathbb{Z}}(\widetilde{\mu_F,\mu_F})$ denote the unique nontrivial extension of $\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F,\mu_F)$ by $\mathbb{Z}/2$ if the characteristic of F is not 2, and $\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F,\mu_F)$ in characteristic 2.

Theorem 2.5. *There is a natural short exact sequence*

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}(\widetilde{\mu_{\mathbb{F}_q},\mu_{\mathbb{F}_q}}) \to \mathrm{H}_3(\mathrm{SL}_2(\mathbb{F}_q),\mathbb{Z}[1/p]) \to \mathcal{B}(\mathbb{F}_q) \to 0$$

for any finite field $\mathbb{F}_q$ of order $q = p^f$.

Furthermore, there is a natural isomorphism

$$\mathrm{H}_3(\mathrm{SL}_2(\mathbb{F}_q),\mathbb{Z}[1/p]) \cong K_3^{\mathrm{ind}}(\mathbb{F}_q).$$

(See [4], Corollary 7.5.)

Furthermore, the calculations in [4], sections 5 and 7, show that

Lemma 2.6.

$$\mathcal{B}(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z}/(q+1)/2, & q \text{ odd} \\ \mathbb{Z}/(q+1), & q \text{ even}\end{cases}$$

and if $K \subset \mathrm{SL}_2(\mathbb{F}_q)$ is a cyclic subgroup of order $(q+1)'$ then the composite map

$$\mathbb{Z}/(q+1)' \cong \mathrm{H}_3(K,\mathbb{Z}[\tfrac{1}{2}]) \to \mathrm{H}_3(\mathrm{SL}_2(\mathbb{F}_q),\mathbb{Z}[\tfrac{1}{2}]) \to \mathcal{B}(\mathbb{F}_q)[\tfrac{1}{2}]$$

is an isomorphism.

2.6. **The map** $\mathrm{H}_3(G,\mathbb{Z}) \to \mathcal{RB}(F)$**.** In section 6 of [4] a recipe is given for calculating the homomorphism $\mathrm{H}_3(G,\mathbb{Z}) \to \mathcal{RB}(F)$ for subgroups G of $\mathrm{SL}_2(F)$. We recall this calculation in the case that G is a finite cyclic subgroup.

First, given 4 distinct points $x_0, x_1, x_2, x_3 \in \mathbb{P}^1(F)$ we define the *refined cross ratio* $\mathrm{cr}(x_0, x_1, x_2, x_3) \in \mathcal{RP}(F)$ by

$$\mathrm{cr}(x_0,x_1,x_2,x_3) = \begin{cases} \left\langle \frac{(x_2-x_0)(x_0-x_1)}{x_2-x_1} \right\rangle \left[\frac{(x_2-x_1)(x_3-x_0)}{(x_2-x_0)(x_3-x_1)} \right], & \text{if } x_i \neq \infty \\ \langle x_1 - x_2 \rangle \left[\frac{x_1-x_2}{x_1-x_3} \right], & \text{if } x_0 = \infty \\ \langle x_2 - x_0 \rangle \left[\frac{x_3-x_0}{x_2-x_0} \right], & \text{if } x_1 = \infty \\ \langle x_0 - x_1 \rangle \left[\frac{x_3-x_0}{x_3-x_1} \right], & \text{if } x_2 = \infty \\ \left\langle \frac{(x_2-x_0)(x_0-x_1)}{x_2-x_1} \right\rangle \left[\frac{x_2-x_1}{x_2-x_0} \right], & \text{if } x_3 = \infty \end{cases}$$

Now suppose that G is a finite cyclic subgroup of $\mathrm{SL}_2(F)$ of order r with generator t. We choose $x \in \mathbb{P}^1(F)$ with trivial stabilizer $G_x = 1$, and choose $y \in \mathbb{P}^1(F) \setminus G \cdot x$.

Lemma 2.7. *The composite*

$$\mathbb{Z}/n \cong \mathrm{H}_3(G,\mathbb{Z}) \to \mathcal{RP}(F)$$

is given by the formula

$$1 \mapsto \sum_{i=0}^{r-1} \mathrm{cr}(\beta_3^{x,y}(1,t,t^{i+1},t^{i+2})).$$

where

$$\begin{aligned} \beta_3^{x,y}(1,t,t,t^2) &= 0 \\ \beta_3^{x,y}(1,t,t^{i+1},t^{i+2}) &= (x, t(x), t^{i+1}(x), t^{i+2}(x)) \textit{ for } 1 \le i \le r-3 \\ \beta_3^{x,y}(1,t,t^{r-1},1) &= (y, t(x), t^{-1}(x), x) - (y, x, t(x), t^{-1}(x)) \\ \beta_3^{x,y}(1,t,t^r,t^{r+1}) &= \beta_3^{x,y}(1,t,1,t) \\ &= \begin{cases} 0, & y = t(y) \\ (y, t(y), x, t(x)) + (y, t(y), t(x), x), & y \neq t(y) \end{cases} \end{aligned}$$

Furthermore, the resulting map is independent of the particular choice of x and y.

3. Some algebra in $\mathcal{RP}(F)$

In this section we study certain key elements and submodules of the refined pre-Bloch group of a field F.

3.1. **The elements** $\psi_i(x)$ **and the modules** $\mathcal{K}_F^{(i)}$**.** We recall the elements

$$\{x\} := [x] + \left[x^{-1}\right] \in \mathcal{P}(F)$$

(for $x \in F^\times$). A straightforward calculation - see Suslin [11] - shows that these symbols allow us to define a group homomorphism

$$F^\times \to \mathcal{P}(F), \quad x \mapsto \{x\}$$

whose kernel contains $(F^\times)^2$;i.e. we have

$$\left\{x^2\right\} = 0 \text{ and } \{xy\} = \{x\} + \{y\} \text{ for all } x, y.$$

In particular, these elements satisfy $2\{x\} = 0$ for all x.

We now consider two liftings of these elements in $\mathcal{RP}(F)$: For $x \in F^\times$ we let

$$\psi_1(x) := [x] + \langle -1\rangle \left[x^{-1}\right]$$

and

$$\psi_2(x) := \begin{cases} \langle 1-x\rangle \left(\langle x\rangle [x] + \left[x^{-1}\right]\right), & x \neq 1 \\ 0, & x = 1 \end{cases}$$

(If $F = \mathbb{F}_2$, we interpret this as $\psi_i(1) = 0$ for $i = 1, 2$. For $F = \mathbb{F}_3$, we have $\psi_1(-1) = \psi_2(-1) = [-1] + \langle -1\rangle [-1]$.)

The maps $F^\times \to \mathcal{RP}(F), x \mapsto \psi_i(x)$ are no longer homomorphisms, but are derivations:

Lemma 3.1. *Let F be a field. For $i \in \{1, 2\}$, the map*

$$F^\times \to \mathcal{RP}(F), x \mapsto \psi_i(x)$$

is a 1*-cocycle; i.e. we have*

$$\psi_i(xy) = \langle x\rangle \psi_i(y) + \psi_i(x) \textit{ for all } x, y \in F^\times.$$

Proof. The statement is trivial for $F = \mathbb{F}_2$ or $\mathbb{F}_3$. We can thus assume F has at least 4 elements. If $x = 1$ or $y = 1$, the required identities are clear. If $x \neq 1$ and $y \neq x^{-1}$ the relation $0 = S_{x,xy} + \langle -1\rangle S_{x^{-1},x^{-1}y^{-1}}$ in $\mathcal{RP}(F)$ yields the identity

$$\psi_1(x) - \psi_1(y) + \langle x\rangle \psi_1(xy) = 0.$$

Thus we must also prove that $\langle x\rangle \psi_1\left(x^{-1}\right) + \psi_1(x) = 0$ for all $x \neq 1$. Fix $x \neq 1$ and choose $y \notin \{1, x^{-1}\}$ (here we use that F has at least 4 elements). Then

$$\langle y\rangle \psi_1(x) = \psi_1(xy) - \psi_1(y) = -\langle xy\rangle \psi_1\left(x^{-1}\right)$$

and multiplying by $\langle y\rangle$ gives the required identity.

Now, for $x, y \in F^\times$, let

$$Q(x, y) := \langle x\rangle \left[\frac{x}{y}\right] + \langle y\rangle \left[\frac{y}{x}\right] \in \mathcal{RP}(F).$$

Then

$$Q(x, y) = \langle y\rangle \left(\left\langle \frac{x}{y}\right\rangle \left[\frac{x}{y}\right] + \left[\frac{y}{x}\right]\right) = \langle y\rangle \left\langle 1 - \frac{x}{y}\right\rangle \psi_2\left(\frac{x}{y}\right) = \langle y - x\rangle \psi_2\left(\frac{x}{y}\right).$$

For $a, b \neq 1$, the relation $0 = S_{a,b} + S_{a,b}$ in $\mathcal{RP}(F)$ gives the identity

$$Q(a^{-1} - 1, b^{-1} - 1) = Q(a^{-1}, b^{-1}) + Q(1 - a, 1 - b).$$

Thus

$$\left\langle b^{-1} - a^{-1}\right\rangle \psi_2\left(\frac{a^{-1} - 1}{b^{-1} - 1}\right) = \left\langle b^{-1} - a^{-1}\right\rangle \psi_2\left(\frac{b}{a}\right) + \langle a - b\rangle \psi_2\left(\frac{1 - a}{1 - b}\right)$$

and hence

$$\psi_2\left(\frac{b^{-1} - 1}{a^{-1} - 1}\right) = \psi_2\left(\frac{b}{a}\right) + \left\langle \frac{b}{a}\right\rangle \psi_2\left(\frac{1 - a}{1 - b}\right).$$

Now if we fix $x, y \neq 1$ with $xy \neq 1$, we can solve the equations

$$x = \frac{b}{a}, \qquad y = \frac{1 - a}{1 - b}$$

for a and b and prove the required identity for $\psi_2(\,)$. □

Corollary 3.2. *For $i \in \{1,2\}$ we have:*

1. $\psi_i\left(xy^2\right) = \psi_i(x) + \psi_i\left(y^2\right)$ *for all x, y*
2. $\langle\langle x\rangle\rangle \psi_i\left(y^2\right) = 0$ *for all x, y*
3. $2 \cdot \psi_i(-1) = 0$ *for all i*
4. $\psi_i\left(x^2\right) = -\langle\langle x\rangle\rangle \psi_i(-1)$ *for all x*
5. $2 \cdot \psi_i\left(x^2\right) = 0$ *for all x and if -1 is a square in F then $\psi_i\left(x^2\right) = 0$ for all x.*
6. $\langle\langle x\rangle\rangle \langle\langle y\rangle\rangle \psi_i(-1) = 0$ *for all x, y*
7. $\langle -1\rangle \langle\langle x\rangle\rangle \psi_i(y) = \langle\langle x\rangle\rangle \psi_i(y)$ *for all x, y*
8. *Let*
$$\epsilon(F) := \begin{cases} 1, & -1 \in (F^\times)^2 \\ 2, & -1 \notin (F^\times)^2 \end{cases}$$
The map $G_F \to \mathcal{RP}(F)$, $\langle x\rangle \mapsto \epsilon(F)\psi_i(x)$ is a well-defined 1-cocycle.

Proof. The identities $\psi_i(1) = 0$ and $\psi_i\left(x^{-1}\right) = \langle -1\rangle \psi_i(x)$ follow from the definition of $\psi_i(\,)$. More generally, let M be an R_F-module and let $\psi : F^\times \to M$ be a 1-cocycle satisfying
$$\psi(1) = 0 \text{ and } \psi(x^{-1}) = \langle -1\rangle \psi(x) \text{ for all } x \in F^\times.$$

1. For all x, y we have
$$\psi(xy^2) = \left\langle y^2\right\rangle \psi(x) + \psi(y^2) = \psi(x) + \psi(y^2)$$
since $\left\langle y^2\right\rangle = 1$ in R_F.
2. The cocycle condition implies that $\langle\langle x\rangle\rangle \psi(y) = \langle\langle y\rangle\rangle \psi x$ for all x, y. Thus
$$\langle\langle x\rangle\rangle \psi(y^2) = \left\langle\left\langle y^2\right\rangle\right\rangle \psi(x) = 0.$$
3. We have $\langle -1\rangle \psi(-1) = \psi(-1)$ and thus
$$0 = \psi(1) = \psi(-1 \cdot -1) = \psi(-1) + \langle -1\rangle \psi(-1) = 2\psi(-1).$$
4. For all x we have
$$\psi(x) = \psi\left(\frac{1}{x} \cdot x^2\right) = \psi\left(\frac{1}{x}\right) + \psi(x^2) = \langle -1\rangle \psi(x) + \psi(x^2).$$
Thus
$$\psi(x^2) = -\langle\langle -1\rangle\rangle \psi(x) = -\langle\langle x\rangle\rangle \psi(-1).$$
5. The first statement follows from (3) and (4). For the second, observe that for any x we have
$$\psi(x^2) = -\langle\langle -1\rangle\rangle \psi(x)$$
and $\langle\langle -1\rangle\rangle = 0$ if -1 is a square.
6. This statement follows from (2) and (4).
7. This is a restatement of (6); namely
$$\langle\langle -1\rangle\rangle \langle\langle x\rangle\rangle \psi(y) = \langle\langle x\rangle\rangle \langle\langle y\rangle\rangle \psi(-1) = 0.$$
8. By (5), $\epsilon(F)\psi(x^2) = 0$ in M for all x and thus $\epsilon(F)\psi(xy^2) = \epsilon(F)\psi(x)$ for all x, y. Thus the proposed map is well-defined (and is thus clearly a 1-cocycle).

□

Now let $\mathcal{K}_F^{(i)}$ denote the R_F-submodule of $\mathcal{RP}(F)$ generated by the set $\{\psi_i(x) \mid x \in F^\times\}$.

Lemma 3.3. *Let F be a field. Then for $i \in \{1, 2\}$*

$$\lambda_1\left(\mathcal{K}_F^{(i)}\right) = \mathrm{p}_{-1}^+(\mathcal{I}_F) \subset \mathcal{I}_F^2$$

and $\mathrm{Ker}(\lambda_1|_{\mathcal{K}_F^{(i)}})$ *is annihilated by* 4.

Proof. We use the identities

$$\langle\langle a\rangle\rangle \langle\langle b\rangle\rangle = \langle\langle ab\rangle\rangle - \langle\langle a\rangle\rangle - \langle\langle b\rangle\rangle, \quad \langle -1\rangle \langle\langle a\rangle\rangle = \langle\langle -a\rangle\rangle - \langle\langle -1\rangle\rangle, \quad \left\langle\left\langle ab^2\right\rangle\right\rangle = \langle\langle a\rangle\rangle$$

in $\mathcal{I}_F$.

Thus

$$\begin{aligned}
\lambda_1(\psi_1(x)) &= \lambda_1([x]) + \langle -1\rangle \lambda_1(\left[x^{-1}\right]) \\
&= \langle\langle x\rangle\rangle \langle\langle 1-x\rangle\rangle + \langle -1\rangle \langle\langle x\rangle\rangle \langle\langle x(x-1)\rangle\rangle \\
&= \langle\langle x(1-x)\rangle\rangle - \langle\langle x\rangle\rangle - \langle\langle 1-x\rangle\rangle + \langle -1\rangle (\langle\langle x-1\rangle\rangle - \langle\langle x\rangle\rangle - \langle\langle x(x-1)\rangle\rangle) \\
&= \langle\langle x(1-x)\rangle\rangle - \langle\langle x\rangle\rangle - \langle\langle 1-x\rangle\rangle + \langle\langle 1-x\rangle\rangle - \langle\langle -x\rangle\rangle - \langle\langle x(1-x)\rangle\rangle + \langle\langle -1\rangle\rangle \\
&= \langle\langle -1\rangle\rangle - \langle\langle x\rangle\rangle - \langle\langle -x\rangle\rangle = \langle\langle -x\rangle\rangle \cdot \langle\langle x\rangle\rangle \\
&= -\mathrm{p}_{-1}^+ \cdot \langle\langle x\rangle\rangle
\end{aligned}$$

Thus $\lambda_1(\mathcal{K}_F^{(1)}) = \mathrm{p}_{-1}^+(\mathcal{I}_F)$.

For $x \neq 1$ we have $\psi_2(x) = \langle x(1-x)\rangle [x] + \langle 1-x\rangle \left[x^{-1}\right]$ and thus

$$\begin{aligned}
\lambda_1(\psi_2(x)) &= \langle x(1-x)\rangle \langle\langle x\rangle\rangle \langle\langle 1-x\rangle\rangle + \langle 1-x\rangle \langle\langle x\rangle\rangle \langle\langle x(x-1)\rangle\rangle \\
&= \langle x(1-x)\rangle (\langle\langle x(1-x)\rangle\rangle - \langle\langle x\rangle\rangle - \langle\langle 1-x\rangle\rangle) + \langle 1-x\rangle (\langle\langle x-1\rangle\rangle - \langle\langle x(x-1)\rangle\rangle - \langle\langle x\rangle\rangle) \\
&= -\langle\langle 1-x\rangle\rangle - \langle\langle x\rangle\rangle + \langle\langle x(1-x)\rangle\rangle + \langle\langle -1\rangle\rangle - \langle\langle -x\rangle\rangle - \langle\langle x(1-x)\rangle\rangle + \langle\langle 1-x\rangle\rangle \\
&= \langle\langle -1\rangle\rangle - \langle\langle x\rangle\rangle - \langle\langle -x\rangle\rangle = \langle\langle x\rangle\rangle \cdot \langle\langle -x\rangle\rangle = -\mathrm{p}_{-1}^+ \cdot \langle\langle x\rangle\rangle .
\end{aligned}$$

Thus $\lambda_1(\mathcal{K}_F^{(2)}) = \mathrm{p}_{-1}^+(\mathcal{I}_F)$ also.

For the second statement, recall that for any group G and any $\mathbb{Z}[G]$-module M a 1-cocycle $\rho : G \to M$ gives rise to $\mathbb{Z}[G]$-homomorphism $\mathcal{I}_G \to M,\ g-1 \mapsto \rho(g)$. Thus, for $i \in \{1, 2\}$, we have a well-defined R_F-homomorphism

$$\mathcal{I}_F \to \mathcal{RP}(F),\ \langle\langle x\rangle\rangle \mapsto 2\psi_i(x) .$$

Combining this with the inclusion $\mathrm{p}_{-1}^+(\mathcal{I}_F) \to \mathcal{I}_F$ we obtain an R_F-module homomorphism $\mu : \mathrm{p}_{-1}^+(\mathcal{I}_F) \to \mathcal{K}_F^{(i)}$ sending $\mathrm{p}_{-1}^+ \langle\langle x\rangle\rangle = \langle\langle x\rangle\rangle + \langle -1\rangle \langle\langle x\rangle\rangle$ to $2\psi_i(x) + \langle -1\rangle 2\psi_i(x) = 4\psi_i(x)$.

It follows that $\mu \circ \left(\lambda_1|_{\mathcal{K}_F^{(i)}}\right)$ is just multiplication by 4, and the result is proved. □

Remark 3.4. Since $\mathrm{p}_{-1}^+\mathcal{I}_F$ is a free abelian group, it follows that, as an abelian group, $\mathcal{K}_F^{(i)}$ decomposes as a direct sum $A \oplus \left(\mathcal{K}_F^{(i)}\right)_{\mathrm{tors}}$ where A is a free abelian group and 4 annihilates $\left(\mathcal{K}_F^{(i)}\right)_{\mathrm{tors}} = \mathrm{Ker}(\lambda_1|_{\mathcal{K}_F^{(i)}})$.

Furthermore, if $\langle\langle -1\rangle\rangle \psi_i(x) = 0$ for all x (for example, if $-1 \in (F^\times)^2$) then $\psi_i\left(x^2\right) = 0$ for all x and the map $G_F \to \mathcal{RP}(F), \langle x\rangle \mapsto \psi_i(x)$ is already a well-defined 1-cocycle. The above arguments then show that $\left(\mathcal{K}_F^{(i)}\right)_{\mathrm{tors}} = \mathrm{Ker}(\lambda_1|_{\mathcal{K}_F^{(i)}})$ is annihilated by 2.

For any field F we will let

$$\widetilde{\mathcal{RP}}(F) := \mathcal{RP}(F)/\mathcal{K}_F^{(1)}.$$

Observe that for any x we have $\Lambda(\psi_1(x)) = -[-x,x] \in \mathrm{RS}^2_{\mathbb{Z}}(F^\times)$ so that $\Lambda(\mathcal{K}^{(1)}_F)$ is the R_F-submodule of $\mathrm{RS}^2_{\mathbb{Z}}(F^\times)$ generated by the symbols $[-x,x]$. Thus we set

$$\widetilde{\mathrm{RS}^2_{\mathbb{Z}}(F^\times)} := \mathrm{RS}^2_{\mathbb{Z}}(F^\times)/\Lambda(\mathcal{K}^{(1)}_F)$$

and let

$$\widetilde{\Lambda} : \widetilde{\mathcal{RP}}(F) \to \widetilde{\mathrm{RS}^2_{\mathbb{Z}}(F^\times)}$$

be the resulting R_F-homomorphism induced by Λ. Finally we let $\widetilde{\mathcal{RB}}(F) := \mathrm{Ker}(\widetilde{\Lambda})$.

Corollary 3.5. *For any field F the natural map $\mathcal{RP}(F) \to \widetilde{\mathcal{RP}}(F)$ induces a surjection $\mathcal{RB}(F) \to \widetilde{\mathcal{RB}}(F)$ whose kernel is annihilated by* 4.

Proof. The surjectivity of the map is clear from the definitions. On the other hand, the kernel of the map $\mathcal{RB}(F) \to \widetilde{\mathcal{RB}}(F)$ is $\mathcal{RB}(F) \cap \mathcal{K}^{(1)}_F$ which is contained in $\mathrm{Ker}(\lambda_1|_{\mathcal{K}^{(1)}_F})$. □

For finite fields, the results of [4] allow us to be more precise:

Lemma 3.6. *Let $\mathbb{F}_q$ be a finite field with q elements.*

(1) *If q is even or if $q \equiv 1 \pmod 4$ then $\mathcal{B}(\mathbb{F}_q) = \mathcal{RB}(\mathbb{F}_q) = \widetilde{\mathcal{RB}}(\mathbb{F}_q)$. This group is cyclic of order $(q+1)$ when q is even and $(q+1)/2$ when $q \equiv 1 \pmod 4$.*
(2) *If $q \equiv 3 \pmod 4$ then $\widetilde{\mathcal{RB}}(\mathbb{F}_q) = \mathcal{RB}(\mathbb{F}_q)/\langle\{-1\}\rangle$ is cyclic of order $(q+1)/4$.*

Proof. The cases $q = 2$ or $q = 3$ are immediate.

When $q \geq 4$, by the results in section 7 of [4], the natural map $\mathcal{RP}(\mathbb{F}_q) \to \mathcal{P}(\mathbb{F}_q)$ induce a natural isomorphism $\mathcal{RB}(\mathbb{F}_q) \cong \mathcal{B}(\mathbb{F}_q)$ and for all x the image of $\psi_1(x) \in \mathcal{RB}(\mathbb{F}_q)$ is $\{x\} \in \mathcal{B}(\mathbb{F}_q)$, which has order divisible by 2. If q is even or if $q \equiv 1 \pmod 4$ then $\mathcal{B}(\mathbb{F}_q)$ is cyclic of order $q+1$ or $(q+1)/2$ and hence is of odd order.

On the other hand, if $q \equiv 3 \pmod 4$ the results of [4] show that $\mathcal{K}^{(1)}_{\mathbb{F}_q} = \langle\psi_1(-1)\rangle$ is cyclic of order 2 and is contained in $\mathcal{RB}(\mathbb{F}_q)$. □

3.2. **The constants C_F and D_F.** In the classical pre-Bloch group $\mathcal{P}(F)$ the expression $[x] + [1-x]$ is known to be independent of $x \in F \setminus \{0,1\}$. Furthermore, this constant has order dividing 6. We consider now an analogous constant in $\mathcal{RP}(F)$.

Let F be a field with at least 4 elements. For $x \in F^\times$, $x \neq 1$ we let

$$\tilde{C}(x) := [x] + \langle -1\rangle [1-x] \text{ and } C(x) = \tilde{C}(x) + \langle\langle 1-x\rangle\rangle \psi_1(x).$$

Lemma 3.7. *Let F be a field with at least 4 elements. Then $C(x)$ is constant; i.e. for all $x, y \in F \setminus \{0,1\}$ we have $C(x) = C(y)$ in $\mathcal{RP}(F)$.*

Proof. In $\mathcal{RP}(F)$ we have

$$\begin{aligned}
0 &= S_{x,y} + \langle -1\rangle S_{1-x,1-y}\\
&= [x] - [y] + \langle x\rangle \left[\frac{y}{x}\right] - \left\langle x^{-1}-1\right\rangle \left[\frac{x^{-1}-1}{y^{-1}-1}\right] + \langle 1-x\rangle \left[\frac{1-x}{1-y}\right]\\
&\quad + \langle -1\rangle [1-x] - \langle -1\rangle [1-y] + \langle x-1\rangle \left[\frac{1-y}{1-x}\right] - \left\langle 1-x^{-1}\right\rangle \left[\frac{y^{-1}-1}{x^{-1}-1}\right] + \langle -x\rangle \left[\frac{x}{y}\right]\\
&= \tilde{C}(x) - \tilde{C}(y) + \langle x\rangle \psi_1\left(\frac{y}{x}\right) - \left\langle 1-x^{-1}\right\rangle \psi_1\left(\frac{1-y^{-1}}{1-x^{-1}}\right) + \langle x-1\rangle \psi_1\left(\frac{y-1}{x-1}\right)\\
&= \left(\tilde{C}(x) - \psi_1(x) + \psi_1\left(1-x^{-1}\right) - \psi_1(x-1)\right) - \left(\tilde{C}(y) - \psi_1(y) + \psi_1\left(1-y^{-1}\right) - \psi_1(y-1)\right)
\end{aligned}$$

(using the cocycle property of $\psi_1\left(\,\right)$ to obtain the last line). Furthermore

$$\begin{aligned}\psi_1\left(1-x^{-1}\right)-\psi_1\left(x-1\right)-\psi_1\left(x\right) &= \psi_1\left((x-1)x^{-1}\right)-\psi_1\left(x-1\right)-\psi_1\left(x\right)\\ =\langle x-1\rangle\,\psi_1\left(x^{-1}\right)-\psi_1\left(x\right) &= \langle 1-x\rangle\,\psi_1\left(x\right)-\psi_1\left(x\right)=\langle\langle 1-x\rangle\rangle\,\psi_1\left(x\right).\end{aligned}$$

□

Definition 3.8. Thus, for a given field F with at least 4 elements, we will denote by C_F the common value of the expressions $C(x)$ for $x\in F\setminus\{0,1\}$.

Furthermore, we let $C_{\mathbb{F}_2}$ denote the distinguished generator of $\mathcal{RP}(\mathbb{F}_2)=\mathcal{RB}(\mathbb{F}_2)$ and we let $C_{\mathbb{F}_3}:=\psi_1\left(-1\right)=(1+\langle -1\rangle)\,[-1]\in\mathcal{RB}(\mathbb{F}_3)$.

Corollary 3.9. *Let F be a field.*

(1) $\langle -1\rangle\, C_F=C_F$
(2) *If F contains a cube root of unity, then $C_F=\psi_1\left(-1\right)$.*

Proof. (1) For $x\notin\{0,1\}$ we have $C_F=C(1-x)=\langle -1\rangle\, C(x)=\langle -1\rangle\, C_F$ (since $\langle -1\rangle\,\langle\langle 1-x\rangle\rangle\,\psi_1\left(x\right)=\langle\langle 1-x\rangle\rangle\,\psi_1\left(x\right)$ by Corollary 3.2 (7)).

(2) Let ζ be a primitive cube root of unity in F and take $x=-\zeta$. Then $1-x=x^{-1}$ and thus

$$\begin{aligned}C_F=C(x)=\psi_1\left(x\right)+\langle\langle x\rangle\rangle\,\psi_1\left(x\right)=\langle x\rangle\,\psi_1\left(x\right)=\langle -1\rangle\,\psi_1\left(x\right)=\psi_1\left(x^{-1}\right)\\ =\psi_1\left(-1\cdot\zeta^2\right)=\psi_1\left(-1\right)+\psi_1\left(\zeta^2\right)=\psi_1\left(-1\right)\end{aligned}$$

since

$$\psi_1\left(\zeta^2\right)=-\langle\langle\zeta\rangle\rangle\,\psi_1\left(-1\right)=0$$

because ζ is a square.

□

Lemma 3.10. *For any $x\in F\setminus\{0,1\}$ we have*

$$2C_F=[x]+\langle -1\rangle\left[\frac{1}{1-x^{-1}}\right]-\psi_1\left(\frac{1}{1-x}\right)$$

Proof.

$$\begin{aligned}2C_F &= C(x)+C\left(\frac{1}{1-x}\right)\\ &= [x]+\langle -1\rangle\,[1-x]+\langle\langle 1-x\rangle\rangle\,\psi_1\left(x\right)+\left[\frac{1}{1-x}\right]+\langle -1\rangle\left[\frac{1}{1-x^{-1}}\right]+\left\langle\!\!\left\langle 1-x^{-1}\right\rangle\!\!\right\rangle\left[\frac{1}{1-x}\right]\\ &= [x]+\langle -1\rangle\left[\frac{1}{1-x^{-1}}\right]+\left(\langle -1\rangle\,\psi_1\left(1-x\right)+\langle\langle 1-x\rangle\rangle\,\psi_1\left(x\right)+\langle\langle x(x-1)\rangle\rangle\,\psi_1\left(1-x\right)\right)\end{aligned}$$

and

$$\begin{aligned}&\langle -1\rangle\,\psi_1\left(1-x\right)+\langle\langle 1-x\rangle\rangle\,\psi_1\left(x\right)+\langle\langle x(x-1)\rangle\rangle\,\psi_1\left(1-x\right)\\ &=\psi_1\left(x-1\right)-\psi_1\left(-1\right)+\psi_1\left(x(1-x)\right)-\psi_1\left(1-x\right)-\psi_1\left(x\right)+\psi_1\left(-x(1-x)^2\right)-\psi_1\left(x(x-1)\right)-\psi_1\left(1-x\right)\\ &=\psi_1\left(x-1\right)-\psi_1\left(-1\right)+\psi_1\left(x(1-x)\right)-\psi_1\left(1-x\right)\\ &\quad-\psi_1\left(x\right)+\psi_1\left(-x\right)+\psi_1\left((1-x)^2\right)-\psi_1\left(x(x-1)\right)-\psi_1\left(1-x\right)\\ &=\left[(1+\langle x\rangle+\langle 1-x\rangle+\langle x(1-x)\rangle)\psi_1\left(-1\right)\right]+\psi_1\left((1-x)^2\right)-\psi_1\left(1-x\right)\end{aligned}$$

(using the identity $\psi_1\left(a\right)-\psi_1\left(-a\right)=\langle a\rangle\,\psi_1\left(-1\right)$).

The first term in the last line is equal to $\langle\langle x\rangle\rangle \langle\langle 1-x\rangle\rangle \psi_1(-1)$ (using $2\cdot\psi_1(-1)=0$), and this is zero by 3.2 (6).

Finally, in general we have
$\psi_1(a) = \psi_1\left(a^2\right) + \psi_1\left(a^{-1}\right)$ and thus

$$\psi_1\left(a^2\right) - \psi_1(a) = -\psi_1\left(\frac{1}{a}\right).$$

□

We denote $2\cdot C_F$ by D_F.

It follows that for all $x \neq 0, 1$ we have

$$D_F = D(x) := [x] + \langle -1\rangle\left[\frac{1}{1-x^{-1}}\right] - \psi_1\left(\frac{1}{1-x}\right).$$

Lemma 3.11. *For any field F we have $3D_F = 6C_F = 0$.*

Proof. Fix $x \neq 0, 1$. Then

$$\begin{aligned}
3D_F &= D\left(\frac{1}{x}\right) + D\left(\frac{1}{1-x^{-1}}\right) + D(1-x)\\
&= \left[\frac{1}{x}\right] + \langle -1\rangle\left[\frac{1}{1-x}\right] - \left[\frac{1}{1-x^{-1}}\right] - \langle -1\rangle\left[1-x^{-1}\right]\\
&+ \left[\frac{1}{1-x^{-1}}\right] + \langle -1\rangle[x] - [1-x] - \langle -1\rangle\left[\frac{1}{1-x}\right]\\
&+ [1-x] + \langle -1\rangle\left[1-x^{-1}\right] - \left[\frac{1}{x}\right] - \langle -1\rangle[x]\\
&= 0.
\end{aligned}$$

□

Remark 3.12. Examples show that this is best possible. Under the natural map $\mathcal{RP}(F) \to \mathcal{P}(F)$ the image of C_F is the constant $C := [x] + [1-x] \in \mathcal{B}(F)$. It can be shown that this element has order 6 for example when $F = \mathbb{R}$ (see [11]) or when F is a finite field with q elements and $q \equiv -1 \pmod{12}$ (see [4]).

Theorem 3.13. *Let F be a field. Then*

(1) *For all $x \in F^\times$,*

$$\langle\langle x\rangle\rangle D_F = \psi_1(x) - \psi_2(x).$$

(2) *Let $E = F(\zeta_3)$. Then*

$$\langle\langle x\rangle\rangle D_F = 0 \text{ if } \pm x \in N_{E/F}(E^\times) \subset F^\times.$$

Proof. We consider first the case of a finite field F. The results in the final section of [4] show that the natural map $\mathcal{RP}(F) \to \mathcal{P}(F)$ induces an isomorphism $\mathcal{RB}(F) \cong \mathcal{B}(F)$. Now $D_F \in \mathcal{RB}(F)$ and thus $\langle\langle x\rangle\rangle D_F = 0$ for all x. Similarly for all x, $\psi_1(x) - \psi_2(x) \in \mathcal{RB}(F)$ and this maps to $\{x\} - \{x\} = 0$ in $\mathcal{B}(F)$.

Thus, we can assume without loss that F is an infinite field.

Let

$$t := \begin{bmatrix} -1 & 1\\ -1 & 0\end{bmatrix} \in \mathrm{SL}_2(F).$$

So

$$t^2 = t^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

and for $x \in \mathbb{P}^1(F)$ we have $t(x) = 1 - x^{-1}$ and $t^{-1}(x) = (1-x)^{-1}$.

We let $\Phi(T)$ be the polynomial $T^2 - T + 1 \in F[T]$.

Thus $t(x) = x$ if and only if $\Phi(x) = 0$ and this happens if and only if $\zeta = \zeta_3 \in F$ and $x = -\zeta$ or $x = -\zeta^{-1}$.

We now choose $x \in F^\times \setminus \{1\}$ with $t(x) \neq x$ and $y \in F^\times \setminus \{1\}$ satisfying $t(y) \neq y$ and $y \notin \{x, t(x), t^{-1}(x)\}$.

By Lemma 2.7, the natural composite map

$$\mathbb{Z}/3 = \mathrm{H}_3(\langle t \rangle, \mathbb{Z}) \to \mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}) \to \mathcal{RB}(F)$$

is given by the formula

$$1 \mapsto \sum_{i=0}^{2} \mathrm{cr}(\beta_3^{x,y}(1, t, t^{i+1}, t^{i+2})) := C(x,y).$$

Thus, using Lemma 2.7 again, we have

$$\begin{aligned}
C(x,y) &= \mathrm{cr}(\beta_3^{x,y}(1,t,t^{-1},1)) + \mathrm{cr}(\beta_3^{x,y}(1,t,1,t)) \\
&= \mathrm{cr}(y, t(x), t^{1}(x), x) - \mathrm{cr}(y, x, t(x), t^{-1}(x)) + \mathrm{cr}(y, t(y), x, t(x)) + \mathrm{cr}(y, t(y), t(x), x) \\
&= \left\langle \frac{(t^{-1}(x)-y)(y-t(x))}{t^{-1}(x)-t(x)} \right\rangle \left[\frac{(t^{-1}(x)-t(x))(x-y)}{(t^{1}(x)-y)(x-t(x))} \right] - \left\langle \frac{(t(x)-y)(y-x)}{t(x)-x} \right\rangle \left[\frac{(t(x)-x)(t^{-1}(x)-y)}{(t(x)-y)(t^{-1}(x)-x)} \right] \\
&\quad + \left\langle \frac{(x-y)(y-t(y))}{x-t(y)} \right\rangle + \left\langle \frac{(t(x)-y)(y-t(y))}{t(x)-t(y)} \right\rangle \left[\frac{(t(x)-t(y))(x-y)}{(t(x)-y)(x-t(y))} \right] \\
&= \langle a \rangle E
\end{aligned}$$

where

$$a = a(x,y) = -\Phi(x)\frac{(x-y)}{x-1-xy}$$

and

$$E = E(x,y) = \langle s \rangle [s] - \langle -1 \rangle \left[\frac{1}{1-s} \right] + \psi_2\left(\frac{s}{1-s^{-1}} \right)$$

with

$$s = s(x,y) = \frac{x-y}{1-y+xy}.$$

Now

$$\langle s-1 \rangle \psi_2\left(s^{-1}\right) = \langle 1-s \rangle \psi_2(s) = \langle s \rangle [s] + \left[s^{-1}\right]$$

and

$$D_F = D\left(s^{-1}\right) = \left[s^{-1}\right] + \langle -1 \rangle \left[\frac{1}{1-s} \right] - \psi_1\left(\frac{1}{1-s^{-1}} \right)$$

so that

$$\begin{aligned}
E &= \langle s-1 \rangle \psi_2\left(s^{-1}\right) - D_F - \psi_1\left(\frac{1}{1-s^{-1}} \right) + \psi_2\left(\frac{s}{1-s^{-1}} \right) \\
&= \psi_2\left(\frac{1}{1-s^{-1}} \right) - \psi_1\left(\frac{1}{1-s^{-1}} \right) - D_F \quad (\text{ since } \langle s-1 \rangle = \left\langle s(1-s^{-1}) \right\rangle).
\end{aligned}$$

Now let

$$r := \frac{1}{1-s^{-1}} = \frac{x-y}{x-1-xy}$$

and observe that $\langle a\rangle = \langle -\Phi(x)r\rangle$.

Thus we have

$$C = C(x,y) = \langle -\Phi(x)r\rangle\left(\psi_2(r) - \psi_1(r) - D_F\right) \in \mathcal{RB}(F) \tag{1}$$

has order dividing 3 and is independent of the choice of x and y. If we fix x satisfying $x^3 \neq 1$, then clearly r can assume any nonzero value other than 1 by appropriate choice of y.

Taking $r = -1$ and multiplying both sides of the equation by 4 we see that

$$C = -\langle -\Phi(x)\rangle D_F = -\langle \Phi(x)\rangle D_F$$

where $x^3 \neq 1$. In particular, it follows that $\langle \Phi(x)\rangle D_F$ is independent of x (with $x^3 \neq 1$). Using the identity

$$\Phi(x)\Phi(y) = (x+y-1)^2\Phi\left(\frac{xy-1}{x+y-1}\right)$$

it follows that $\langle \Phi(x)\rangle D_F = D_F$ whenever $x^3 \neq 1$. Thus more generally

$$\left\langle\!\left\langle \pm\Phi(x)z^2\right\rangle\!\right\rangle D_F = 0$$

for $x^3 \neq 1$ and any z. Statement (2) of the theorem follows immediately.

Now choose $r = b^2$ for some $b \neq \pm 1$ in formula (1). This gives

$$C = \langle \Phi(x)\rangle\left(\psi_2\left(b^2\right) - \psi_1\left(b^2\right)\right) - D_F.$$

Multiplying both sides by 4 again shows that $C = -D_F$.

Using this, formula (1) says that

$$\langle\!\langle -\Phi(x)r\rangle\!\rangle D_F = \langle -\Phi(x)r\rangle\left(\psi_2(r) - \psi_1(r)\right)$$

or, equivalently,

$$\langle\!\langle -\Phi(x)r\rangle\!\rangle D_F = \psi_1(r) - \psi_2(r)$$

for all r. Since $\langle\!\langle -\Phi(x)\rangle\!\rangle D_F = 0$ for all x, it follows that

$$\langle\!\langle r\rangle\!\rangle D_F = \psi_1(r) - \psi_2(r)$$

for all $r \in F^\times$, proving statement (1) of the theorem. □

Remark 3.14. We will see below that statement (2) is in general best possible. For example if F is a local or global field not containing ζ_3 then often we have $\langle\!\langle x\rangle\!\rangle D_F \neq 0$ (and hence $\psi_1(x) \neq \psi_2(x)$) when x is not a norm from $F(\zeta_3)$.

Remark 3.15. Observe that the element t chosen in the last proof lies in the image of $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(F)$.

Now it is well known that $\mathrm{SL}_2(\mathbb{Z})$ can be expressed as an amalgamated product $C_4 *_{C_2} C_6$. Here C_4 is cyclic of order 4 with generator a and C_6 is cyclic of order 6 with generator b with

$$a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

Thus $b^2 = t$ is our matrix of order 3. A straightforward direct calculation, using this decomposition, shows that $\mathrm{H}_3(\mathrm{SL}_2(\mathbb{Z}),\mathbb{Z})$ is cyclic of order 12 and that the inclusion $G := \langle t\rangle \to \mathrm{SL}_2(\mathbb{Z})$ induces an isomorphism $\mathrm{H}_3(G,\mathbb{Z}) \cong \mathrm{H}_3(\mathrm{SL}_2(\mathbb{Z}),\mathbb{Z})[3]$.

It follows that for any field F, the image of the natural map $\mathrm{H}_3(\mathrm{SL}_2(\mathbb{Z}),\mathbb{Z})[3] \to \mathcal{RB}(F)[3]$ is the cyclic subgroup generated by $-D_F$.

For any ring A, we will let $\mathbb{D} := \mathbb{D}_A$ denote the image of the generator of $\mathrm{H}_3(G,\mathbb{Z})$ in $\mathrm{H}_3(\mathrm{SL}_2(A),\mathbb{Z})$. Thus if A is a subring of the field F, $\mathbb{D}_A$ maps to $-D_F$ under the map $\mathrm{H}_3(\mathrm{SL}_2(A),\mathbb{Z}) \to \mathcal{RB}(F)$.

4. Valuations and Specialization

In this section, we prove the existence of specialization or reduction maps from the refined pre-Bloch group of a field to the refined pre-Bloch group of the residue field of a valuation. We use these specialization maps to obtain lower bounds for the groups $\mathcal{I}_F\mathcal{RB}(F)$ for fields with valuations.

4.1. **Some preliminary definitions.** By Theorem 3.13, for any field F we have an identity of R_F-submodules of $\mathcal{RP}(F)$

$$\mathcal{K}^{(1)}_F + \mathcal{K}^{(2)}_F = \mathcal{K}^{(1)}_F + \mathcal{I}_F D_F.$$

We let $\mathcal{K}_F$ denote the module $\mathcal{K}^{(1)}_F + \mathcal{K}^{(2)}_F$.

Lemma 4.1. *For any field F we have*

$$\mathcal{K}_F = \mathcal{K}^{(1)}_F + \mathcal{I}_F D_F = \mathcal{K}^{(1)}_F + \mathcal{I}_F C_F$$

and

$$\mathcal{K}_F \cong \mathcal{K}^{(1)}_F \oplus \mathcal{I}_F D_F.$$

Proof. Theorem 3.13 immediately implies that $\mathcal{K}_F = \mathcal{K}^{(1)}_F + \mathcal{I}_F D_F$. On the other hand since

$$\begin{aligned} D_F &\equiv \left[\frac{1}{x}\right] + \langle -1\rangle \left[\frac{1}{1-x}\right] \pmod{\mathcal{K}^{(1)}_F} \\ C_F &\equiv [x] + \langle -1\rangle [1-x] \pmod{\mathcal{K}^{(1)}_F} \\ \text{and } \left[x^{-1}\right] &\equiv -\langle -1\rangle [x] \pmod{\mathcal{K}^{(1)}_F}, \end{aligned}$$

it follows that $C_F \equiv -D_F \pmod{\mathcal{K}^{(1)}_F}$ and thus

$$\mathcal{K}^{(1)}_F + \mathcal{I}_F C_F = \mathcal{K}^{(1)}_F + \mathcal{I}_F D_F.$$

Finally, suppose that $a \in \mathcal{K}^{(1)}_F \cap \mathcal{I}_F D_F$. Then $3a = 0$ since $3D_F = 0$ and $4a = 0$ since $a \in \mathcal{K}^{(1)}_F \cap \mathcal{RB}(F)$, so that $a = 0$. Thus $\mathcal{K}^{(1)}_F \cap \mathcal{I}_F D_F = 0$. □

For any field F, we now define

$$\widehat{\mathcal{RP}(F)} := \mathcal{RP}(F)/\mathcal{K}_F = \widetilde{\mathcal{RP}}(F)/\mathcal{I}_F C_F \text{ and } \widehat{\mathcal{RB}(F)} = \widetilde{\mathcal{RB}}(F)/\mathcal{I}_F C_F.$$

Corollary 4.2. *For any field F there is natural short exact sequence*

$$0 \to \mathcal{I}_F D_F \to \widetilde{\mathcal{RB}}(F) \to \widehat{\mathcal{RB}(F)} \to 0.$$

If F is a finite field the action of G_F on $\mathcal{RB}(F) = \mathcal{B}(F)$ is trivial and thus we have

Corollary 4.3. *For any finite field F, we have $\widehat{\mathcal{RB}(F)} = \widetilde{\mathcal{RB}}(F)$.*

Now let Γ be an ordered (additive) abelian group and let $v : F^\times \to \Gamma$ be a valuation, which we assume to be surjective. As usual, let

$$\mathcal{O} = \mathcal{O}_v = \{0\} \cup \{x \in F^\times \mid v(x) \geq 0\}$$

be the valuation ring, let

$$\mathcal{M} = \mathcal{M}_v = \{0\} \cup \{x \in F^\times \mid v(x) > 0\}$$

be the maximal ideal, let $k = k_v = \mathcal{O}/\mathcal{M}$ be the residue field and

$$U = U_v = \{x \in F^\times \mid v(x) = 0\} = \mathcal{O} \setminus \mathcal{M}.$$

Also set

$$U_1 = U_{1,v} = 1 + \mathcal{M} = \mathrm{Ker}(U \to k, u \mapsto \bar{u} := u + \mathcal{M}).$$

For any abelian group A, we let $A/2$ denote $A/2 = A \otimes \mathbb{Z}/2$. Since Γ is a torsion-free group we have a short exact sequence of $\mathbb{F}_2$-vector spaces

$$1 \to U/2 \to G_F \to \Gamma/2 \to 0. \tag{2}$$

We have homomorphisms of commutative rings

$$\begin{tikzcd} \mathbb{Z}[U/2] \arrow[r, hook] \arrow[d] & \mathrm{R}_F \\ \mathrm{R}_k & \end{tikzcd}$$

Thus, if M is any R_k module

$$M_F = M_{F,v} := \mathrm{R}_F \otimes_{\mathbb{Z}[U/2]} M$$

is an R_F-module.

4.2. **The specialization homomorphisms.**

Theorem 4.4. *Let F be a field with valuation v and corresponding residue field k. Then there is a surjective R_F-module homomorphism*

$$\begin{array}{rcl} S := S_v : \mathcal{RP}(F) & \to & \widehat{\mathcal{RP}(k)}_F \\ [a] & \mapsto & \left\{ \begin{array}{rl} 1 \otimes [\bar{a}], & v(a) = 0 \\ 1 \otimes C_k, & v(a) > 0 \\ -(1 \otimes C_k), & v(a) < 0 \end{array} \right. \end{array}$$

Proof. Let Z_1 denote the set of symbols of the form $[x], x \neq 1$ and let $T : \mathrm{R}_F[Z_1] \to \widehat{\mathcal{RP}(k)}_F$ be the unique R_F-homomorphism given by

$$[a] \mapsto \left\{ \begin{array}{rl} 1 \otimes [\bar{a}], & v(a) = 0 \\ 1 \otimes C_k, & v(a) > 0 \\ -(1 \otimes C_k), & v(a) < 0 \end{array} \right.$$

We must prove that $T(S_{x,y}) = 0$ for all $x, y \in F^\times \setminus \{1\}$.

We divide the proof into several cases:

Case (i): $v(x) = v(y) \neq 0$

Subcase (a): $v(x) > 0$.

Then $1 - x, 1 - y \in U_1$ and hence

$$w := \frac{1-y}{1-x} \in U_1.$$

Now

$$u := \frac{y}{x} \in U \qquad \text{and} \qquad \frac{1-y^{-1}}{1-x^{-1}} = \frac{w}{u} \equiv u^{-1} \pmod{U_1}.$$

Thus

$$\begin{aligned} T(S_{x,y}) &= (1\otimes C_k) - (1\otimes C_k) + \langle x\rangle \otimes [\bar{u}] + \langle x\rangle \langle x-1\rangle \otimes \left[\bar{u}^{-1}\right] \\ &= \langle x\rangle \otimes \psi_1(\bar{u}) = 0 \end{aligned}$$

(since $\overline{x-1} = -1$ in $k^\times$).

Subcase (b): $v(x) < 0$.
Then $v(x^{-1}) = v(y^{-1}) > 0$ and

$$w := \frac{1-y^{-1}}{1-x^{-1}} \in U_1.$$

So

$$u := \frac{y}{x} \in U \qquad \text{and } \frac{1-y}{1-x} = uw \equiv u \pmod{U_1}.$$

Thus

$$T(S_{x,y}) = -(1\otimes C_k) + (1\otimes C_k) + \langle x\rangle \otimes [\bar{u}] - \langle x-1\rangle \otimes [\bar{u}] = 0$$

since $\langle x-1\rangle = \langle x\rangle \left\langle 1-x^{-1}\right\rangle$ and $1-x^{-1} \in U_1$.

Case (ii): $v(x), v(y) \neq 0$ and $v(x) \neq v(y)$.

Subcase (a) $v(x) > v(y) > 0$:
Then

$$T(S_{x,y}) = (1\otimes C_k) - (1\otimes C_k) - \langle x\rangle \otimes C_k + \langle x\rangle \otimes C_k = 0.$$

Subcase (b) $v(x) > 0 > v(y)$:
Then

$$T(S_{x,y}) = (1\otimes C_k) + (1\otimes C_k) - \langle x\rangle \otimes C_k + \langle x\rangle \otimes C_k + 1\otimes C_k = 0$$

since $3C_k = 3D_k = 0$ in $\widehat{\mathcal{RP}(k)}$.

Subcase (c) $0 > v(x) > v(y)$:
Then

$$\begin{aligned} T(S_{x,y}) &= -(1\otimes C_k) + (1\otimes C_k) + \langle x\rangle \otimes C_k - \langle 1-x\rangle \otimes C_k \\ &= \langle x\rangle \otimes (C_k - \langle -1\rangle C_k) = 0 \end{aligned}$$

(using $\langle 1-x\rangle = \langle x\rangle \left\langle x^{-1}-1\right\rangle$ and $\overline{x^{-1}-1} = -1$ in $k^\times$).

Subcases (d),(e),(f) where $v(y) > v(x)$ are similar.

Case (iii): $x, y \in U_1$

Subcase (a): $v(1-x) \neq v(1-y)$
Then

$$\frac{y}{x} \in U_1 \qquad \text{and } v\left(\frac{1-y}{1-x}\right) = v\left(\frac{1-y^{-1}}{1-x^{-1}}\right) \neq 0$$

so that

$$T(S_{x,y}) = \pm\left(\left\langle x^{-1}-1\right\rangle \otimes C_k - \langle 1-x\rangle \otimes C_k\right) = 0$$

since $x^{-1}-1 \equiv 1-x \pmod{U_1}$.

Subcase (b): $v(1-x) = v(1-y)$.
Then

$$u := \frac{1-y}{1-x} \in U \qquad \text{and } \frac{1-y^{-1}}{1-x^{-1}} = u\cdot\frac{x}{y} \equiv u \pmod{U_1}.$$

Thus

$$T(S_{x,y}) = -\langle x-1\rangle \otimes [\bar{u}] + \langle x(x-1)\rangle \otimes [\bar{u}] = 0$$

since $x \in U_1$.

Case (iv): $x \in U_1$, $v(y) \neq 0$. Then

$$v\left(\frac{1-x^s}{1-y^s}\right) > 0 \text{ for } s \in \{\pm 1\}$$

and hence

$$T(S_{x,y}) = \epsilon(-(1 \otimes C_k) + (1 \otimes C_k)) - \langle 1-x\rangle \otimes (C_k - C_k) = 0$$

where $\epsilon = \mathrm{sign}(v(y)) \in \{\pm 1\}$.

Case (v): $x \in U \setminus U_1$ and $v(y) \neq 0$.

Thus $v(x) = v(1-x) = v(1-x^{-1}) = 0$.

Subcase (a): $v(y) > 0$.

Then

$$v(y) = v\left(\frac{y}{x}\right) > 0 \text{ and } 1-y \in U_1.$$

Furthermore,

$$v\left(\frac{1-x^{-1}}{1-y^{-1}}\right) = -v(1-y^{-1}) = v(y) > 0.$$

Thus

$$\begin{aligned} T(S_{x,y}) &= 1 \otimes [\bar{x}] - (1 \otimes C_k) + (1 \otimes C_k) - (1 \otimes C_k) + 1 \otimes \langle 1-\bar{x}\rangle [1-\bar{x}] \\ &= 1 \otimes (-C_k + [\bar{x}] + \langle 1-\bar{x}\rangle [1-\bar{x}]) \end{aligned}$$

But for any $a \in k^\times$ we have

$$\langle a\rangle [a] \equiv -\left\langle a^{-1}\right\rangle \equiv \langle -1\rangle [a] \pmod{\mathcal{K}_k}$$

since $\psi_1(a) = \psi_2(a) = 0$ in $\mathcal{K}_k$.

Thus

$$T(S_{x,y}) = 1 \otimes (-C_k + C_k) = 0 \in \widehat{\mathcal{RP}(k)}_F.$$

Subcase (b): $v(y) < 0$

Then

$$v(y) = v\left(\frac{y}{x}\right) < 0 \text{ and } v\left(\frac{1-x}{1-y}\right) = -v(y) > 0.$$

So

$$\begin{aligned} T(S_{x,y}) &= 1 \otimes [\bar{x}] + (1 \otimes C_k) - (1 \otimes C_k) - 1 \otimes \left\langle \bar{x}^{-1} - 1\right\rangle \left[1-\bar{x}^{-1}\right] + (1 \otimes C_k) \\ &= 1 \otimes (C_k + [\bar{x}] - \left\langle \bar{x}^{-1} - 1\right\rangle \left[1-\bar{x}^{-1}\right]) = 0 \end{aligned}$$

since

$$[\bar{x}] \equiv -\langle -1\rangle \left[\bar{x}^{-1}\right] \pmod{\mathcal{K}_k}.$$

Case (vi): $y \in U \setminus U_1$ and $v(x) \neq 0$.

Subcase (a): $v(x) > 0$

We have $1-x \in U_1$ and hence

$$\frac{1-x}{1-y} \equiv (1-y)^{-1} \pmod{U_1}.$$

Now

$$v\left(\frac{y}{x}\right) < 0 \text{ and } v\left(\frac{1-x^{-1}}{1-y^{-1}}\right) = v\left(\frac{y}{x}\right) < 0.$$

so

$$\begin{aligned} T(S_{x,y}) &= (1\otimes C_k) - 1\otimes[\bar{y}] - \langle x\rangle\otimes C_k + \langle x\rangle\otimes C_k - 1\otimes\left[\frac{1}{1-\bar{y}}\right] \\ &= 1\otimes\left(C_k - [\bar{y}] - \left[\frac{1}{1-\bar{y}}\right]\right) \\ &= 1\otimes(C_k - [\bar{y}] - \langle -1\rangle[1-\bar{y}]) = 0 \in \widehat{\mathcal{RP}(k)}_F. \end{aligned}$$

Subcase (b): $v(x) < 0$.
So $1-x^{-1}\in U_1$ and

$$\begin{aligned} T(S_{x,y}) &= -(1\otimes C_k) - 1\otimes[\bar{y}] + \langle x\rangle\otimes C_k - \langle -1\rangle\otimes\left[\frac{1}{1-\bar{y}^{-1}}\right] - \langle x\rangle\otimes C_k \\ &= -(1\otimes(C_k + [\bar{y}] + \langle -1\rangle\left[\frac{1}{1-\bar{y}^{-1}}\right] \\ &= -(1\otimes\left(C_k + \langle -1\rangle(\left[\bar{y}^{-1}\right] + \left[1-\bar{y}^{-1}\right])\right) = 0. \end{aligned}$$

Case (vii): $x\in U\setminus U_1$ and $y\in U_1$.
Then

$$-v(1-y) = v\left(\frac{1-x}{1-y}\right) = v\left(\frac{1-x^{-1}}{1-y^{-1}}\right) < 0$$

and hence

$$T(S_{x,y}) = 1\otimes([\bar{x}] + \langle\bar{x}\rangle\left[\bar{x}^{-1}\right] + C_k - C_k) = 0$$

since $\psi_2(\bar{x}) = 0$ in $\widehat{\mathcal{RP}(k)}$.

Case (viii): $x, y\in U\setminus U_1$.
Then $1-x, 1-y, 1-x^{-1}, 1-y^{-1}\in U\setminus U_1$ and thus

$$T(S_{x,y}) = 1\otimes S_{\bar{x},\bar{y}} = 0 \text{ in } \mathcal{RP}(k)_F.$$

□

Given a valuation v on a field F and an element $a\in F^\times$ with $v(a)\neq 0$, we will let $\epsilon_v(a)$ denote $\mathrm{sign}(v(a))\in\{\pm 1\}$.

Corollary 4.5. *Let $\rho : \mathrm{R}_F\to\mathrm{R}_k$ be any ring homomorphism satisfying $\rho(\langle u\rangle) = \langle\bar{u}\rangle$ if $u\in U$. Then ρ induces a R_F-module structure on $\mathcal{RP}(k)$ and there is a surjective homomorphism of R_F-modules $S = S_\rho : \mathcal{RP}(F)\to\widehat{\mathcal{RP}(k)}$ determined by*

$$[a] \quad\mapsto\quad \begin{cases} [\bar{a}], & a\in U_v \\ \epsilon_v(a)C_k, & a\notin U_v \end{cases}$$

Proof. We compose S with the surjective map

$$\widehat{\mathcal{RP}(k)}_F \to \widehat{\mathcal{RP}(k)},\ x\otimes y\mapsto \rho(x)y.$$

□

Now, if we choose a splitting, $j : \Gamma/2\to G_F$ of the sequence (2), we obtain a ring isomorphism

$$\mathrm{R}_F\cong\mathbb{Z}[U/2\times j(\Gamma/2)]\cong\mathbb{Z}[U/2]\otimes_{\mathbb{Z}}\mathbb{Z}[\Gamma/2].$$

On the other hand, any character $\chi : \Gamma/2\to\mu_2 = \{1,-1\}$ induces a ring homomorphism $\mathbb{Z}[\Gamma/2]\to\mathbb{Z}$. Thus, given a splitting j and a character χ, we obtain a ring homomorphism

$$\rho = \rho_{j,\chi} : \mathrm{R}_F\to\mathrm{R}_k, \langle u\cdot j(\gamma)\rangle\mapsto\chi(\gamma)\langle\bar{u}\rangle$$

and a corresponding surjective specialization map

$$S = S_{j,\chi} : \mathcal{RP}(F) \to \widehat{\mathcal{RP}(k)}$$

which is also an R_F-homomorphism.

Example 4.6. For example, if v is a discrete valuation, then $\Gamma = \mathbb{Z}$ and $\Gamma/2 = \mathbb{Z}/2$. Any choice, π, of uniformizing parameter, determines a splitting $\mathbb{Z}/2 \to G_F,\ 1 \mapsto \langle \pi \rangle$. There are two characters, ϵ, on $\mathbb{Z}/2$, namely 1 and -1. Thus we get ring homomorphisms

$$\rho_{\pi,\epsilon} : \mathrm{R}_F \to \mathrm{R}_k, \langle u\pi^r \rangle \mapsto \epsilon^r \langle \bar{u} \rangle .$$

and corresponding specialization homomorphisms

$$S_{\pi,\epsilon} : \mathcal{RP}(F) \to \widehat{\mathcal{RP}(k)}.$$

In general, the specialization homomorphisms $S_{j,\chi}$ do not restrict to homomorphisms $\mathcal{RB}(F) \to \widehat{\mathcal{RB}(k)}$ of Bloch groups.

To see this, we begin with the following observation:

Lemma 4.7. *For a field F and $x \in F^\times$ we have*

$$\mathrm{p}^+_{-1} \langle\langle x \rangle\rangle \widetilde{\mathcal{RP}}(F) \subset \widetilde{\mathcal{RB}}(F).$$

Proof. Since G_F acts trivially on $\mathrm{S}^2_{\mathbb{Z}}(F^\times)$ we have $\langle\langle x \rangle\rangle \mathrm{S}^2_{\mathbb{Z}}(F^\times) = 0$ and thus

$$\begin{aligned} \widetilde{\Lambda}(\langle\langle x \rangle\rangle \mathrm{p}^+_{-1} [a]) &= (\langle\langle x \rangle\rangle \langle\langle 1-a \rangle\rangle \mathrm{p}^+_{-1} \langle\langle a \rangle\rangle , 0) \\ &= (\langle\langle x \rangle\rangle \langle\langle 1-a \rangle\rangle \langle\langle -a \rangle\rangle \langle\langle a \rangle\rangle , 0) \\ &= \langle\langle x \rangle\rangle \langle\langle 1-a \rangle\rangle [-a, a] = 0 \in \mathrm{R}\widetilde{\mathrm{S}^2_{\mathbb{Z}}(F^\times)}. \end{aligned}$$

□

Corollary 4.8. *Let F be a field with valuation $v : F^\times \to \Gamma$, with corresponding residue field k. Let $j : \Gamma/2 \to G_F$ be a splitting, and let χ be a* nontrivial *character on* $\Gamma/2$.

Then the image of

$$S_{j,\chi}|_{\mathcal{RB}(F)} : \mathcal{RB}(F) \to \widehat{\mathcal{RP}(k)}$$

contains $2\mathrm{p}^+_{-1}\widehat{\mathcal{RP}(k)}$.

Proof. First observe that $S_{j,\chi}$ induces a homomorphism

$$\widetilde{\mathcal{RP}}(F) \to \widehat{\mathcal{RP}(k)}$$

since

$$S_{j,\chi}(\psi_1(x)) = \begin{cases} \psi_1(\bar{x}), & v(x) = 0 \\ 0, & v(x) \neq 0 \end{cases}$$

Suppose that $\gamma \in \Gamma$ with $\chi(\gamma) = -1$ and let $\pi := j(\gamma)$. Since $\widehat{\mathcal{RP}(k)}$ is an R_F-module via the homomorphism $\rho_{j,\chi}$, $\langle \pi \rangle$ acts on $\widehat{\mathcal{RP}(k)}$ as multiplication by -1. Thus $\langle\langle \pi \rangle\rangle \widehat{\mathcal{RP}(k)} = 2\widehat{\mathcal{RP}(k)}$ and hence $S_{j,\chi}$ induces a surjective homomorphism

$$\langle\langle \pi \rangle\rangle \mathrm{p}^+_{-1}\widetilde{\mathcal{RP}}(F) \to 2\mathrm{p}^+_{-1}\widehat{\mathcal{RP}(k)}.$$

□

Remark 4.9. As just note, the maps $S_{j,\chi} : \mathcal{RP}(F) \to \widehat{\mathcal{RP}(k)}$ descend to maps $\widetilde{\mathcal{RP}}(F) \to \widehat{\mathcal{RP}(k)}$. However, they do not usually induce maps $\widehat{\mathcal{RP}(F)} \to \widehat{\mathcal{RP}(k)}$. In fact, below we will use the homomorphisms $S_{j,\chi}$ to detect the non-triviality of $\mathcal{I}_F C_F$ for global and (some) local fields.

Corollary 4.10. *Let F be a field with valuation $v : F^\times \to \Gamma$ and* algebraically closed *residue field k. Let $j : \Gamma/2 \to G_F$ be a splitting, and let $\chi : \Gamma/2 \to \mu_2$ be a* nontrivial *character. Then $S_{j,\chi}$ induces a surjective map*

$$\mathcal{RB}(F) \twoheadrightarrow \mathcal{RP}(k).$$

Proof. Since k is quadratically closed, $\mathcal{RP}(k) = \mathcal{P}(k)$. Furthermore, since k is algebraically closed $\psi_i(x) = \{x\} = 0$ for all x. Thus $\widehat{\mathcal{RP}(k)} = \mathcal{P}(k)$. Thus $\mathrm{p}^+_{-1}\widehat{\mathcal{RP}(k)} = 2\mathcal{P}(k)$. However, since k is algebraically closed, $\mathcal{P}(k)$ is divisible and thus $2\mathcal{P}(k) = \mathcal{P}(k)$. □

Corollary 4.11. *Let F be a field with valuation $v : F^\times \to \Gamma$ and corresponding residue field k. Let $j : \Gamma/2 \to G_F$ be a splitting, and let $\chi : \Gamma/2 \to \mu_2$ be a* nontrivial *character. Choose $\gamma \in \Gamma$ with $\chi(\gamma) = -1$ and let $\pi = j(\gamma)$.*

Then the map $S_{j,\chi}$ induces a surjective homomorphism of $\mathrm{R}_F[\frac{1}{2}]$-modules

$$\mathrm{e}^-_\pi\mathrm{e}^+_{-1}\left(\mathcal{RB}(F)[\tfrac{1}{2}]\right) \to \mathrm{e}^+_{-1}\left(\widehat{\mathcal{RP}(k)}[\tfrac{1}{2}]\right)$$

In particular, if $-1 \in (F^\times)^2$, then there is a surjective homomorphism

$$\mathrm{e}^-_\pi\left(\mathcal{RB}(F)[\tfrac{1}{2}]\right) \to \widehat{\mathcal{RP}(k)}[\tfrac{1}{2}]$$

Proof. Since

$$\mathrm{e}^-_\pi\mathrm{e}^+_{-1}\left(\widetilde{\mathcal{RP}}(F)[\tfrac{1}{2}]\right) \subset \widetilde{\mathcal{RB}}(F)[\tfrac{1}{2}]$$

by the results above, it follows that

$$\mathrm{e}^-_\pi\mathrm{e}^+_{-1}\left(\widetilde{\mathcal{RP}}(F)[\tfrac{1}{2}]\right) = \mathrm{e}^-_\pi\mathrm{e}^+_{-1}\left(\widetilde{\mathcal{RB}}(F)[\tfrac{1}{2}]\right) \cong \mathrm{e}^-_\pi\mathrm{e}^+_{-1}\left(\mathcal{RB}(F)[\tfrac{1}{2}]\right)$$

□

Remark 4.12. In general, the specialization maps $S_{j,\chi}$ depend on the choice of splitting j and character χ.

We can free ourselves of the dependence on the choice of splitting j (or the choice of uniformizer π, in the case of a discrete valuation) by replacing the target $\widehat{\mathcal{RP}(k)}$ with

$$\widehat{\mathcal{P}(k)} := \widehat{\mathcal{RP}(k)}_{k^\times} = \frac{\mathcal{P}(k)}{\mathcal{S}_k}$$

where $\mathcal{S}_k$ denotes the (2-torsion) subgroup of $\mathcal{P}(k)$ generated by the elements $\{x\}$.

Since $k^\times$, and hence $U/2$, acts trivially on $\mathcal{P}(k)$, we get well-defined surjective specialization maps

$$S_\chi : \mathcal{RP}(F) \to \widehat{\mathcal{P}(k)}$$

which depend only on the choice of character $\chi : \Gamma/2 \to \mu_2$.

It is important to note that although $\widehat{\mathcal{P}(k)}$ is a trivial R_k-module, the R_F-module structure induced from a character χ is not usually trivial.

Thus, if χ is not the trivial character and if $\pi \in G_F$ satsifies $\chi(v(\pi)) = -1$, then π acts as multiplication by -1 on $\widehat{\mathcal{P}(k)}$ and S_χ induces a surjective homomorphism

$$\mathrm{e}^-_\pi\left(\mathcal{RB}(F)[\tfrac{1}{2}]\right) \to \widehat{\mathcal{P}(k)}[\tfrac{1}{2}]$$

5. Some applications

From the existence of these specialization maps it follows that if F is field with many valuations, then $\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z})_0$ must be large. For example:

Theorem 5.1. *Let F be a field and let $\mathcal{V}$ be a family of discrete values on F satisfying*

(1) *For any $x \in F^\times$, $v(x) = 0$ for all but finitely many $v \in \mathcal{V}$*
(2) *The map*
$$F^\times \to \oplus_{v\in\mathcal{V}}\mathbb{Z}/2, \quad a \mapsto \{v(a)\}_v$$
is surjective.

Then there is a natural surjective homomorphism
$$\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}])_0 \to \bigoplus_{v\in\mathcal{V}}\widehat{\mathcal{P}(k_v)}[\tfrac{1}{2}].$$

Proof. We have
$$\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}])_0 = \mathcal{RB}(F)_0[\tfrac{1}{2}] = \mathcal{I}_F\mathcal{RB}(F)[\tfrac{1}{2}] = \sum_{a\in G_F} \mathrm{e}^-_a\mathcal{RB}(F)[\tfrac{1}{2}]$$
by Lemma 2.3.

We denote by S_v the specialization map $\mathcal{RP}(F) \to \widehat{\mathcal{P}(k_v)}$ corresponding to the character $\epsilon = -1$. Note that if $a \in F^\times$ with $v(a) \equiv 0 \pmod 2$, then for any $y \in \mathcal{RP}(F)$, $S_v(\mathrm{e}^-_a y) = \mathrm{e}^-_a S_v(y) = 0$ since $\langle\bar{a}\rangle$ acts trivially on $\mathcal{P}(k_v)$. Thus, for any $a \in F^\times$, $y \in \mathcal{RP}(F)$, we have $S_v(\mathrm{e}^-_a y) = 0$ for almost all v. It follows that for any $x \in \mathcal{I}_F\mathcal{RB}(F)[\tfrac{1}{2}]$, $S_v(x) = 0$ for almost all $v \in \mathcal{V}$.

Thus the maps S_v induce a well-defined $\mathrm{R}_F[\tfrac{1}{2}]$-homomorphism
$$S : \mathcal{RB}(F)[\tfrac{1}{2}]_0 \to \oplus_v\widehat{\mathcal{P}(k_v)}[\tfrac{1}{2}], \qquad x \mapsto \{S_v(x)\}_v.$$

Finally, let $y = \{y_v\}_v \in \oplus_v\widehat{\mathcal{P}(k_v)}[\tfrac{1}{2}]$. Let $T = \{v \in \mathcal{V} \mid y_v \neq 0\}$. For each $v \in T$, choose $x_v \in \mathcal{RB}(F)[\tfrac{1}{2}]$ with $S_v(x_v) = y_v$.

For each $v \in T$, choose $\pi_v \in F^\times$ satisfying $w(\pi_v) \equiv \delta_{w,v} \pmod 2$ for all $w \in \mathcal{V}$. Then
$$y = S\left(\sum_{v\in T} \mathrm{e}^-_{\pi_v}\cdot x_v\right).$$
□

Since $\widehat{\mathcal{P}(k)}[\tfrac{1}{2}] = \mathcal{P}(k)[\tfrac{1}{2}] = \mathcal{B}(k)[\tfrac{1}{2}]$ for a finite field k, we deduce:

Corollary 5.2. *Let F be a global field and let S be a finite set of places with the property that $\mathrm{Cl}(O_S)/2 = 0$. Let $\mathcal{V}$ be the set of all finite places of F not in S. Then there is a surjective homomorphism of R_F-modules*
$$\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}])_0 \to \bigoplus_{v\in\mathcal{V}}\mathcal{B}(k_v)[\tfrac{1}{2}].$$

Here, if π_v is a uniformizer for v, then the square class of π_v acts as -1 of the factor $\mathcal{B}(k_v)[\tfrac{1}{2}]$ on the right.

Remark 5.3. Note that it follows that the groups $\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z})$ are not finitely generated, for any global field F. By contrast, when F is a global field, the groups $\mathrm{H}_3(\mathrm{SL}_3(F),\mathbb{Z}) \cong K_3^{\mathrm{ind}}(F)$ are well-known to be finitely generated. (Furthermore the stabilization map $\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}) \to \mathrm{H}_3(\mathrm{SL}_3(F),\mathbb{Z})$ is known to be surjective in this case, by the results of [5].)

Remark 5.4. Observe that in the case $F = \mathbb{Q}$ the short exact sequence

$$0 \to \mathrm{H}_3(\mathrm{SL}_2(\mathbb{Q}), \mathbb{Z}[\tfrac{1}{2}])_0 \to \mathrm{H}_3(\mathrm{SL}_2(\mathbb{Q}), \mathbb{Z}[\tfrac{1}{2}]) \to K_3^{\mathrm{ind}}(\mathbb{Q})[\tfrac{1}{2}] \to 0$$

is split, since the composite

$$\mathrm{H}_3(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z}[\tfrac{1}{2}]) \to \mathrm{H}_3(\mathrm{SL}_2(\mathbb{Q}), \mathbb{Z}[\tfrac{1}{2}]) \to K_3^{\mathrm{ind}}(\mathbb{Q})[\tfrac{1}{2}] = \mathcal{B}(\mathbb{Q})[\tfrac{1}{2}]$$

is an isomorphism (of groups of order 3). Thus there is a natural surjective homomorphism

$$\mathrm{H}_3(\mathrm{SL}_2(\mathbb{Q}), \mathbb{Z}[\tfrac{1}{2}]) \twoheadrightarrow K_3^{\mathrm{ind}}(\mathbb{Q})[\tfrac{1}{2}] \oplus \left(\bigoplus\nolimits_p \mathcal{B}(\mathbb{F}_p)[\tfrac{1}{2}]\right)$$

When k is algebraically closed, $\widehat{\mathcal{P}(k)} = \mathcal{P}(k) = \mathcal{RP}(k)$ and $\mathcal{P}(k)$ is uniquely divisible – i.e. a $\mathbb{Q}$-vector space – by the results of Suslin ([11]).

Corollary 5.5. *Let k be an algebraically closed field and let X be an algebraic variety over k, with rational function field $k(X)$. Suppose that X is regular in codimension 1 and that $\mathrm{Cl}(X)/2 = 0$. Let V be the set of codimension 1 subvarieties.*

The theorem says that there is a surjective homomorphism of $\mathrm{R}_{k(X)}$-modules

$$\mathrm{H}_3(\mathrm{SL}_2(k(X)), \mathbb{Q})_0 \to \oplus_{v \in V} \mathcal{P}(k)$$

Remark 5.6. Note that this shows that $\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z})_0$ may have a large torsion-free part in general.

The specialization maps can also detect the nontriviality of elements the third homology of certain arithmetic groups:

Theorem 5.7. *Let F be a number field. Let S be a nonempty set of finite primes of F such that $\mathrm{Cl}^S(F)/2 = 0$, where $\mathrm{Cl}^S(F)$ denotes the subgroup of $\mathrm{Cl}(\mathcal{O}_F)$ generated by the primes of S.*

Then there is a natural surjective map of $\mathbb{F}_3[\mathcal{O}_S^\times/2]$-modules

$$\mathrm{H}_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})[3] \to \oplus_{v \in S} \mathcal{B}(k_v)[3].$$

Proof. For $v \in S$, let R_v denote the composite

$$\mathrm{H}_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z}) \to \mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}) \to \mathcal{RB}(F) \to \widehat{\mathcal{RP}(k_v)}.$$

By [4], section 7, $\mathcal{B}(k)[3] = \widehat{\mathcal{RP}(k)}[3]$ is generated by D_k for any finite field k.

Our hypothesis on $\mathrm{Cl}^S(F)$ guarantees that the map

$$\mathcal{O}_S^\times \to \oplus_{v \in S} \mathbb{Z}/2$$

is surjective, and thus for each $w \in S$, there exists $u_w \in \mathcal{O}_S^\times$ satisfying $v(u_W) \equiv \delta_{v,w} \pmod 2$ for all $v \in S$.

Finally, for each $w \in S$, let

$$\mathbb{D}_{w,S} := \mathrm{e}^-_{u_w} \cdot \mathbb{D}_{\mathcal{O}_S} \in \mathrm{H}_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})[3]$$

(see Remark 3.15) and hence

$$R_v(\mathbb{D}_{w,S}) = -\mathrm{e}^-_{u_w} \cdot (D_{k_v}) = -\delta_{v,w} D_{k_v}.$$

□

.

Corollary 5.8. *Let F be a number field not containing ζ_3. Let S be a finite set of primes of F for which $\mathrm{Cl}^S(F)/2 = 0$. Let $r(S) = |\{v \in S \mid |k_v| \equiv -1 \pmod 3\}|$. Then*

$$3\text{-}rank\,(\mathrm{H}_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})) \geq 1 + r(S).$$

Proof. The 3-rank of $\oplus_{v\in S}\mathcal{B}(k_v)[3]$ is $r(S)$ since $\mathcal{B}(k_v)$ is cyclic of order $(q_v+1)/2$ or q_v+1 where $q_v = |k_v|$ by the results of [4]. On the other hand, letting u_w be as above, the element

$$\left(\prod_{w\in S} \mathrm{e}^+_{u_w}\right)\mathbb{D}_{O_S} \in \mathrm{H}_3(\mathrm{SL}_2(O_S),\mathbb{Z})[3]$$

lies in the kernel of the map

$$\oplus_v R_v : \mathrm{H}_3(\mathrm{SL}_2(O_S),\mathbb{Z})[3] \to \oplus_{v\in S}\mathcal{B}(k_v)[3]$$

but, for any v, maps to $-D_{k_v}$ under the map $S_{v,1} : \mathcal{RB}(F) \to \widehat{\mathcal{P}(k_v)}$, and thus has order 3. □

6. The third homology of SL_2 of local fields

In this section, we use the properties of the refined Bloch group to calculate $\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z})$ up to 2-torsion when F is a local field with finite residue field of odd characteristic (Theorem 6.14).

6.1. **Preliminary results.**

Lemma 6.1. *Let F be a local field with finite residue field k of order p^f. If $\mathbb{Q}_3 \subset F$ then we suppose that $v_F(3)$ and f are both odd. Let $E = F(\zeta_3)$. Then*

(1) $\langle\langle a\rangle\rangle D_F = 0$ *if and only if* $a \in \langle -1\rangle \cdot N_{E/F}(E^\times)$.
(2) *For any uniformizer, π, of F the specialization map $S_{\pi,-1} : \mathcal{RP}(F) \to \widehat{\mathcal{P}(k)}$ induces an isomorphism of R_F-modules*

$$\mathcal{I}_F \cdot D_F \cong \mathbb{Z}\cdot D_k = \mathcal{B}(k)[3].$$

Proof. We have already seen that if $a \in \langle -1\rangle \cdot N_{E/F}(E^\times)$ then $\langle\langle a\rangle\rangle D_F = 0$ (Theorem 3.13).

If $\mathbb{Q}_3 \subset F$, then the conditions stated are precisely those needed to guarantee that $-1 \notin N_{E/F}(E^\times)$. Thus it follows that $\langle -1\rangle N_{E/F}(E^\times) = F^\times$ and thus $\mathcal{I}_F D_F = 0$. Since $\mathcal{B}(k)[3]$ in this case also, the result is trivially true.

For the rest of the proof, we can suppose that $\mathbb{Q}_3 \not\subset F$. Then E/F is unramified and every unit is a norm. Thus $-1 \in N_{E/F}(E^\times)$. If $\zeta_3 \in F$, then $\langle\langle a\rangle\rangle D_F = 0$ for all a, and also $D_k = 0$. The result holds again for trivial reasons.

Otherwise, $F^\times/N_{E/F}(E^\times)$ is cyclic of order 2 generated by a uniformizer π. Also, $q \not\equiv 1 \pmod 3$ so $3|q+1$ and D_k has order 3. $S_{\pi,-1}(D_F) = D_k \neq 0$ and since π acts as -1 on the right, $S_{\pi,-1}(\langle\langle \pi\rangle\rangle D_F) = -2D_k = D_k \neq 0$, so that the statements of the lemma follow immediately. □

Remark 6.2. Of course, if $\mathbb{Q}_3 \subset F$ and $-1 \in N_{E/F}(E^\times)$ then our results do not rule out the possibility that $\mathcal{I}_F D_F$ has order 3 (rather than 0), but this module, if nontrivial, cannot be detected by $S_{\pi,-1}$ (since $D_k = 0$ in this case).

The following lemma will be central to our computations below:

Lemma 6.3. *In any field F we have*

$$\mathrm{e}^-_b\,[a] = 0 \textit{ in } \widehat{\mathcal{RP}(F)}[\tfrac{1}{2}]$$

whenever $a, b \in F^\times$ with $a \equiv -b \pmod{(F^\times)^2}$.

Proof. Let $a \in F^\times$, $a \neq 1$. Then $[a] + \langle -1\rangle\left[a^{-1}\right] = \psi_1(a) = 0$ in $\widehat{\mathcal{RP}(F)}$ and hence $\left[a^{-1}\right] = -\langle -1\rangle [a]$ in $\widehat{\mathcal{RP}(F)}$. But $\langle a\rangle [a] + \left[a^{-1}\right] = \langle 1-a\rangle \psi_2(a) = 0$ also in $\widehat{\mathcal{RP}(F)}$. Thus $\langle -a\rangle [a] = [a]$ and hence $\mathrm{e}^-_{-a}[a] = 0$.

Of course, e^-_x only depends on the square class of x, so the result follows. □

For the remainder of this section we let F be a local field complete with respect to the discrete valuation v, with residue field k of order $q = p^f$, with p *odd*. Furthermore, for simplicity, we will suppose that if $\mathbb{Q}_3 \subset F$ then both $v_F(3)$ and f are odd.

We let $G = G_F$ be the group of square classes. Thus G has order 4: if π is a uniformizing parameter and if u is a nonsquare unit $G = \{1, \langle \pi \rangle, \langle u \rangle, \langle u\pi \rangle\}$. (If $q \equiv 3 \pmod 4$, we can take $u = -1$.)

We denote let $\widehat{G}$ denote the group of characters $\mathrm{Hom}(G, \mu_2) = \mathrm{Hom}(G, \{\pm 1\})$ of G. Given $\chi \in \widehat{G}$, we have the associated idempotent

$$\mathrm{e}_\chi = \frac{1}{4} \sum_{g \in G} \chi(g) \langle g \rangle .$$

Observe that if χ is a *nontrivial* character on G, then

$$\mathrm{e}_\chi = \prod_{a \in \chi^{-1}(-1)} \mathrm{e}_a^- \in \mathrm{R}_F[\tfrac{1}{2}].$$

If M is an $\mathrm{R}_F[\frac{1}{2}]$-module and if $\chi \in \widehat{G}$, then $M_\chi := \mathrm{e}_\chi(M)$ is a submodule and $g \cdot m = \chi(g)m$ for all $g \in G$. Observe that the functor $M \to M_\chi$ is an exact functor on the category of $\mathrm{R}_F[\frac{1}{2}]$-modules.

For the rest of this section we fix the following: Let π be a uniformizing parameter for F and let u be a fixed nonsquare unit, which we take to be -1 in the case $q \equiv 3 \pmod 4$. Clearly a nontrivial character in $\widehat{G}$ is determined by $\chi^{-1}(-1)$. We label the four characters as follows: χ_1 is the trivial character, χ is the character with $\chi^{-1}(-1) = \{\langle \pi \rangle, \langle u\pi \rangle\}$, ψ is the character with $\psi^{-1}(-1) = \{\langle u\pi \rangle, \langle u \rangle\}$ and ψ' is the remaining character.

Note that for any $\mathrm{R}_F[\frac{1}{2}]$-module M, we have a decomposition

$$M = M_{\chi_1} \oplus M_\chi \oplus M_\psi \oplus M_{\psi'}$$

where

$$M_{\chi_1} = M^G \cong M_G = \mathrm{H}_0(F^\times, M)$$

and

$$\mathcal{I}_F M = M_\chi \oplus M_\psi \oplus M_{\psi'}.$$

Since $\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F)$ is a trivial G-module, from the exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F)[\tfrac{1}{2}] \to \mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\tfrac{1}{2}]) \to \mathcal{RB}(F)[\tfrac{1}{2}] \to 0$$

it follows that

$$\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\tfrac{1}{2}])_\rho = \mathcal{RB}(F)[\tfrac{1}{2}]_\rho$$

for any $\rho \neq \chi_1$.

On the other hand,

$$\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\tfrac{1}{2}])_{\chi_1} = \mathrm{H}_0(F^\times, \mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\tfrac{1}{2}])) = K_3^{\mathrm{ind}}(F)[\tfrac{1}{2}].$$

Thus we have a $\mathrm{R}_F[\frac{1}{2}]$-module decomposition

$$\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\tfrac{1}{2}]) \cong K_3^{\mathrm{ind}}(F)[\tfrac{1}{2}] \oplus \mathcal{RB}(F)[\tfrac{1}{2}]_\chi \oplus \mathcal{RB}(F)[\tfrac{1}{2}]_\psi \oplus \mathcal{RB}(F)[\tfrac{1}{2}]_{\psi'}.$$

Our goal in the remainder of this section is to show that the last two factors on the right are zero and that the second factor is isomorphic, via $S_{\pi,-1}$, to $\mathcal{B}(k)[\frac{1}{2}]$.

In order to do this, we make a couple of reductions.

First note that by Corollary 3.5 the map $\mathcal{RB}(F) \to \widetilde{\mathcal{RB}}(F)$ induces an isomorphism of $\mathrm{R}_F[\frac{1}{2}]$-modules

$$\mathcal{RB}(F)[\tfrac{1}{2}] \cong \widetilde{\mathcal{RB}}(F)[\tfrac{1}{2}].$$

Next we let

$$\overline{\mathcal{RP}(F)} := \frac{\mathcal{RP}(F)}{\mathcal{K}_F^{(1)} + \mathrm{R}_F D_F} = \frac{\widetilde{\mathcal{RP}}(F)}{\mathrm{R}_F D_F} = \frac{\widehat{\mathcal{RP}(F)}}{\mathbb{Z}\cdot D_F}$$

and

$$\overline{\mathcal{RB}(F)} := \frac{\widetilde{\mathcal{RB}}(F)}{\mathrm{R}_F D_F} = \frac{\widetilde{\mathcal{RB}}(F)}{\mathcal{D}_F}.$$

where $\mathcal{D}_F = \mathrm{R}_F D_F$.

Lemma 6.4. *Let F be as stated. Then*

(1) $(\mathcal{D}_F)_\psi = 0 = (\mathcal{D}_F)_{\psi'}$.
(2) $(\mathcal{D}_F)_\chi = \mathcal{I}_F D_F$.

Proof. Note that

$$\mathcal{I}_F D_F = (\mathcal{D}_F)_\chi \oplus (\mathcal{D}_F)_\psi \oplus (\mathcal{D}_F)_{\psi'}.$$

So we must prove that the last two factors are 0.

Recall that $\mathrm{e}_\psi = \mathrm{e}_u^- \mathrm{e}_{u\pi}^-$ and $\mathrm{e}_{\psi'} = \mathrm{e}_u^- \mathrm{e}_\pi^-$. Now $\mathrm{e}_u^- = -\langle\langle u\rangle\rangle/2$ and our conditions guarantee that $u \in \langle -1\rangle N_{E/F}(E^\times)$ so that $\langle\langle u\rangle\rangle D_F = 0$ by Theorem 3.13. □

Corollary 6.5. $\mathcal{RB}(F)[\frac{1}{2}]_\psi = \overline{\mathcal{RB}(F)}[\frac{1}{2}]_\psi$ *and* $\mathcal{RB}(F)[\frac{1}{2}]_{\psi'} = \overline{\mathcal{RB}(F)}[\frac{1}{2}]_{\psi'}$.

Now let $\mathcal{S}_k$ be the subgroup of $\mathcal{P}(k)$ generated by the elements $\{x\}$ and let

$$\overline{\mathcal{P}(k)} := \frac{\mathcal{P}(k)}{\mathbb{Z}D_k + \mathcal{S}_k} = \frac{\widehat{\mathcal{P}(k)}}{\mathbb{Z}D_k}.$$

Then $S_{\pi,-1}$ induces a well-defined surjective homomorphism

$$\bar{S}_{\pi,-1} : \overline{\mathcal{RP}(F)} \to \overline{\mathcal{P}(k)}.$$

Lemma 6.6. *The homomorphism*

$$S_{\pi,-1} : \mathcal{RB}(F)[\tfrac{1}{2}]_\chi \to \widehat{\mathcal{P}(k)}[\tfrac{1}{2}]$$

is an isomorphism if and only if the homomorphism

$$\bar{S}_{\pi,-1} : \overline{\mathcal{RB}(F)}[\tfrac{1}{2}]_\chi \to \overline{\mathcal{P}(k)}[\tfrac{1}{2}]$$

is an isomorphism.

Proof. Using Lemma 6.4 (2), we have a commutative diagram of R_F-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_F D_F \arrow[r] \arrow[d, "\cong"] & \mathcal{RB}(F)[\frac{1}{2}]_\chi \arrow[r] \arrow[d, "S_{\pi,-1}"] & \overline{\mathcal{RB}(F)}[\frac{1}{2}]_\chi \arrow[r] \arrow[d, "\bar{S}_{\pi,-1}"] & 0 \\
0 \arrow[r] & \mathbb{Z}D_k \arrow[r] & \widehat{\mathcal{P}(k)}[\frac{1}{2}] \arrow[r] & \overline{\mathcal{P}(k)}[\frac{1}{2}] \arrow[r] & 0
\end{tikzcd}$$

in which the left vertical arrow is an isomorphism by Lemma 6.1 (2). □

6.2. S_3**-dynamics.** In $\overline{\mathcal{RP}(F)}$ we have $D_F = 0$ and $\psi_1(x) = 0$ for all x, so that

$$\left[x^{-1}\right] = [1-x] = -\langle -1\rangle [x].$$

It follows that if $\alpha[x] = 0$ for some $\alpha \in \mathrm{R}_F$, then $\alpha\left[x^{-1}\right] = \alpha[1-x] = 0$ also.

For any field L, the transformations $\sigma, \tau : \mathbb{P}^1(L) \to \mathbb{P}^1(L)$, $\sigma(x) = x^{-1}$ and $\tau(x) = 1-x$ determine an action of the symmetric group on 3 letters S_3. $\{0, 1, \infty\}$ is an orbit for this action.

Thus, more generally, if $\alpha[x] = 0$ in $\overline{\mathcal{RP}(F)}$ then $\alpha[y] = 0$ if $y \in \mathbb{P}^1(F) \setminus \{0, 1, \infty\}$ is in the S_3-orbit of x.

Let $\bar{R} = (k^\times)^2$ and let $\bar{N} = k^\times \setminus (k^\times)^2$. Let

$$\bar{R}_1 = \{a \in \bar{R} \setminus \{1\} | \tau(a) \in \bar{R}\}, \qquad \bar{R}_{-1} = \{a \in \bar{R} | \tau(a) \in \bar{N}\}.$$

Let

$$\bar{N}_1 = \{a \in \bar{N} | \tau(a) \in \bar{R}\}, \qquad \bar{N}_{-1} = \{a \in \bar{N} | \tau(a) \in \bar{N}\}.$$

Thus we have a partition

$$k^\times = \{1\} \cup \bar{R}_1 \cup \bar{R}_{-1} \cup \bar{N}_1 \cup \bar{N}_{-1}$$

Let U be the group of units of F and let U_1 be the kernel of the reduction map $U \to k^\times$. Then ther is a corresponding partition

$$U = U_1 \cup R_1 \cup R_{-1} \cup N_1 \cup N_{-1}$$

where $R_1 = \{a \in U | \bar{a} \in \bar{R}_1\}$ etc.

Lemma 6.7. *Let k be a finite field with q elements.*

(1) *If* $q \equiv 1 \pmod 4$ *then*

$$\bar{R}_1 = \sigma(\bar{R}_1) = \tau(\bar{R}_1)$$
$$\sigma(\bar{N}_1) = \bar{N}_{-1} \textit{ and } \sigma(\bar{R}_{-1}) = \bar{R}_{-1}$$

Furthermore

$$|\bar{R}_{-1}| = |\bar{N}_{-1}| = |\bar{N}_1| = \frac{q-1}{4} \textit{ and } |\bar{R}_1| = \frac{q-5}{4}.$$

(2) *If* $q \equiv 3 \pmod 4$ *then*

$$\bar{N}_{-1} = \sigma(\bar{N}_{-1}) = \tau(\bar{N}_{-1})$$
$$\sigma(\bar{R}_1) = \bar{R}_{-1} \textit{ and } \sigma(\bar{N}_1) = \bar{N}_1$$

Furthermore

$$|\bar{R}_{-1}| = |\bar{N}_1| = |\bar{R}_1| = \frac{q-3}{4} \textit{ and } |\bar{N}_{-1}| = \frac{q+1}{4}.$$

Proof. (1) Clearly $\tau(\bar{R}_1) = \bar{R}_1$ by definition. Furthermore, if $a \in \bar{R}_1$ then

$$1 - \frac{1}{a} = \frac{a-1}{a} = (-1) \cdot (1-a) \cdot \frac{1}{a} \in \bar{R}$$

and thus $a^{-1} = \sigma(a) \in \bar{R}_1$.

It follows that $\sigma(\bar{R}_{-1}) = \bar{R}_{-1}$ since $\sigma(\bar{R}) = \bar{R}$.

Similarly, if $a \in \bar{N}_1$ then $a^{-1} \in \bar{N}$ $1 - a \in \bar{R}$ and thus

$$1 - \frac{1}{a} = (-1) \cdot (1-a) \cdot \frac{1}{a} \in N$$

so that $\sigma(\bar{N}_1) = \bar{N}_{-1}$.

The second statement follows by simple counting since

$$|\bar{N}_1| + |\bar{N}_{-1}| = |\bar{N}| = (q-1)/2.$$

(2) The argument is similar except that this time

$$a - 1 = (-1)\cdot(1-a) \in \begin{cases} \bar{N}, & 1-a \in \bar{R} \\ \bar{R}, & 1-a \in \bar{N} \end{cases}$$

since $-1 \in \bar{N}$.

$\square$

Corollary 6.8. *Let F be a local field with finite residue field of order q.*

(1) *If* $q \equiv 1 \pmod 4$ *then*

$$R_1 = \sigma(R_1) = \tau(R_1)$$
$$\sigma(N_1) = N_{-1} \text{ and } \sigma(R_{-1}) = R_{-1}$$

(2) *If* $q \equiv 3 \pmod 4$ *then*

$$N_{-1} = \sigma(N_{-1}) = \tau(N_{-1})$$
$$\sigma(R_1) = R_{-1} \text{ and } \sigma(N_1) = N_1$$

We will need the following elementary result below:

Lemma 6.9. *Let k be a finite field of odd order, and let $n \in \bar{N}$. Then there exist $r_1, r_2 \in \bar{R}$ such that $n = r_1 + r_2$.*

Proof. If $q \equiv 1 \pmod 4$, there exists $r \in \bar{R}$ with $rn \in \bar{N}_1$. Thus $1 - rn = s \in \bar{R}$ and

$$n = \frac{1}{r} + \left(-\frac{1}{s}\right) = r_1 + r_2.$$

If $q \equiv 3 \pmod 4$ then there exists $r \in \bar{R}$ with $rn \in \bar{N}_{-1}$. Then $1 - nr = m \in \bar{N}$ and thus

$$n = \frac{1}{r} + \left(-\frac{m}{r}\right) = r_1 + r_2.$$

$\square$

6.3. **The main result.** Let $M(F)$ denote the $\mathrm{R}_F[\frac{1}{2}]$-module $e^-_{-1}\mathcal{I}_F[\frac{1}{2}]$.

Lemma 6.10. (1) *If $\rho \neq \chi_1$ then there is a natural short exact sequence of* $\mathrm{R}_F[\frac{1}{2}]$*-modules*

$$0 \longrightarrow \overline{\mathcal{RB}(F)}[\tfrac{1}{2}]_\rho \longrightarrow \overline{\mathcal{RP}(F)}[\tfrac{1}{2}]_\rho \xrightarrow{\bar{\lambda}_1} M(F)_\rho \longrightarrow 0.$$

(2) *If* $q \equiv 1 \pmod 4$ *and if* $\rho \neq \chi_1$ *then*

$$\overline{\mathcal{RB}(F)}[\tfrac{1}{2}]_\rho = \overline{\mathcal{RP}(F)}[\tfrac{1}{2}]_\rho.$$

(3) *If* $q \equiv 3 \pmod 4$ *then*

$$\overline{\mathcal{RB}(F)}[\tfrac{1}{2}]_\chi = \overline{\mathcal{RP}(F)}[\tfrac{1}{2}]_\chi.$$

Proof. There is an exact sequence

$$0 \longrightarrow \overline{\mathcal{RB}(F)} \longrightarrow \overline{\mathcal{RP}(F)} \xrightarrow{\Lambda} \widetilde{\mathrm{RS}^2_{\mathbb{Z}}(F^\times)}$$

where

$$\widetilde{\mathrm{RS}^2_{\mathbb{Z}}(F^\times)} = \frac{\mathrm{RS}^2_{\mathbb{Z}}(F^\times)}{\langle [x,-x] \rangle}.$$

The kernel of the surjection $\mathrm{RS}^2_{\mathbb{Z}}(F^\times) \to \mathcal{I}_F^2$ is a trivial R_F-module. Furthermore, since $\mathcal{I}_F/\mathcal{I}_F^2 \cong G_F$ is a 2-torsion group, we have $\mathcal{I}_F^2[\frac{1}{2}] = \mathcal{I}_F[\frac{1}{2}]$. Putting these facts together we obtain an isomorphism of R_F-modules

$$\mathrm{RS}^2_{\mathbb{Z}}(F^\times)[\tfrac{1}{2}]_\rho \cong \mathcal{I}_F[\tfrac{1}{2}]_\rho$$

for all $\rho \neq \chi_1$. This in turn induces an isomorphism

$$\widetilde{\mathrm{RS}^2_{\mathbb{Z}}(F^\times)}[\tfrac{1}{2}]_\rho \cong \left(\frac{\mathcal{I}_F[\frac{1}{2}]}{\langle \langle\langle x\rangle\rangle \langle\langle -x\rangle\rangle \rangle}\right)_\rho = \left(\frac{\mathcal{I}_F[\frac{1}{2}]}{\mathrm{p}^+_{-1}\mathcal{I}_F[\frac{1}{2}]}\right)_\rho = \left(\frac{\mathcal{I}_F[\frac{1}{2}]}{\mathrm{e}^+_{-1}\mathcal{I}_F[\frac{1}{2}]}\right)_\rho \cong \mathrm{e}^-_{-1}\mathcal{I}_F[\tfrac{1}{2}]_\rho = M(F)_\rho$$

and the homomorphism

$$\overline{\mathcal{RP}(F)}[\tfrac{1}{2}]_\rho \to \widetilde{\mathrm{RS}^2_{\mathbb{Z}}(F^\times)}[\tfrac{1}{2}]_\rho \cong M(F)_\rho$$

is induced by the map $\bar{\lambda}_1 : \overline{\mathcal{RP}(F)} \to M(F)$

$$[x] \mapsto \mathrm{e}^-_{-1}\lambda_1([x]) = \mathrm{e}^-_{-1} \langle\langle x\rangle\rangle \langle\langle 1-x\rangle\rangle .$$

To see that this homomorphism is surjective, observe first that if $q \equiv 1 \pmod 4$, then $\langle -1\rangle = 1$ and hence $\mathrm{e}^-_{-1} = 0$ in $\mathrm{R}_F[\frac{1}{2}]$. Thus $M(F) = 0$ in this case, and there is nothing to prove. This also implies statement (2) of the lemma.

So we can suppose that $q \equiv 3 \pmod 4$. Then, as a $\mathbb{Z}[\frac{1}{2}]$-module, $\mathcal{I}_F[\frac{1}{2}]$ is free of rank 3 with basis e^-_{-1}, e^-_π and $\mathrm{e}^-_{-\pi}$. Thus $\mathrm{e}^-_{-1}(\mathcal{I}_F[\frac{1}{2}])$ is a free $\mathcal{I}_F[\frac{1}{2}]$-module of rank 2 with basis e^-_{-1} and $\mathrm{e}^-_\pi\mathrm{e}^-_{-1}$ (since, for example,$\mathrm{e}^-_{-1}\mathrm{e}^-_{-\pi} = -(\mathrm{e}^-_{-1} + \mathrm{e}^-_\pi\mathrm{e}^-_{-1})$.) A s an $\mathrm{R}_F[\frac{1}{2}]$-module, it is thus generated by e^-_{-1}.

Now let $n \in N_{-1}$. Then $1-n \in N_{-1}$ and

$$\lambda_1\left(\frac{1}{4}[n]\right) = \frac{1}{4}\langle\langle n\rangle\rangle \langle\langle 1-n\rangle\rangle = \frac{1}{4}\langle\langle -1\rangle\rangle \langle\langle -1\rangle\rangle = (\mathrm{e}^-_{-1})^2 = \mathrm{e}^-_{-1}$$

so that the map is surjective.

Finally, observe also that if $q \equiv 3 \pmod 4$ then $\mathrm{e}_\chi\mathrm{e}^-_{-1} = 0$ so that $M(F)_\chi = 0$ and statement (3) also follows. □

Lemma 6.11. *Let F be a local field with finite residue field k of odd order. Then the* $\mathrm{R}_F[\frac{1}{2}]$-*module homomorphism*

$$S_{\pi,-1} : \mathcal{RP}(F)[\tfrac{1}{2}]_\chi \to \widehat{\mathcal{P}(k)}[\tfrac{1}{2}]$$

is an isomorphism.

Proof. By Lemma 6.6 it is enough to prove that the map

$$S_{\pi,-1} : \overline{\mathcal{RP}(F)}[\tfrac{1}{2}]_\chi \to \overline{\mathcal{P}(k)}[\tfrac{1}{2}]$$

is an isomorphism.

For $a \in F^\times$ we will denote $\mathrm{e}_\chi([a]) \in \widehat{\mathcal{RP}(F)}[\frac{1}{2}]$ by $[a]_\chi$.

To prove the result we will show that there is a well-defined $\mathrm{R}_F[\frac{1}{2}]$-module homomorphism

$$T : \overline{\mathcal{P}(k)}[\tfrac{1}{2}] \to \overline{\mathcal{RP}(F)}[\tfrac{1}{2}]_\chi, \qquad T([\bar{a}]) = [a]_\chi$$

(where $a \in U$ maps to $\bar{a} \in k^\times$) which is inverse to $S_{\pi,-1}$.

To show that T is well-defined we must show that $[au]_\chi = [a]$ whenever $a \in U$ and $u \in U_1$.

Since clearly

$$\bar{S}_{\pi,-1} \circ T = \mathrm{Id}_{\widehat{\mathcal{RB}(k)}[\frac{1}{2}]},$$

it is then only necessary to show that T is surjective: we must show that $[a]_\chi = 0$ whenever $v(a) \neq 0$.

Note, to begin with, that since $\mathrm{e}_\chi = \mathrm{e}^-_\pi \mathrm{e}^-_{u\pi}$, we have

$$[a]_\chi = 0 \text{ if } a \equiv -\pi, -u\pi \pmod{(F^\times)^2}$$

by Corollary 6.3. This is clearly equivalent to

$$[a]_\chi = 0 \text{ if } a \equiv \pi, u\pi \pmod{(F^\times)^2}.$$

Thus $[a]_\chi = 0$ if a is of the form $w\pi^{2n-1}$ with $w \in U$ and $n \in \mathbb{Z}$.

Next, recall that if $[a]_\chi = 0$ in $\overline{\mathcal{RP}(F)}_\chi$ for some $a \in F^\times$, then $[b]_\chi = 0$ for any b lying in the S_3-orbit of a.

Thus for any $n \geq 1$, $[a]_\chi = 0$ if $a \in \tau(U\pi^{2n-1}) = 1 - U\pi^{2n-1} \subset U_1$.

Consider now the case $a = w\pi^{2n}$, $w \in U$ and $n \geq 1$. Then we have

$$0 = (S_{1/\pi, a/\pi})_\chi = \left[\frac{1}{\pi}\right]_\chi - \left[\frac{a}{\pi}\right]_\chi + \langle\pi\rangle [a]_\chi - \langle\pi - 1\rangle \left[\pi^{2n-1}\frac{1-\pi}{\pi^{2n-1} - w}\right]_\chi + \langle -\pi\rangle \left[-\frac{1}{\pi} \cdot \frac{1-\pi}{w\pi^{2n-1} - 1}\right]_\chi = \langle\pi\rangle [a]_\chi$$

since all terms other than the third belong to the square classes $\langle\pi\rangle$ or $\langle u\pi\rangle$.

Thus we deduce that $[a]_\chi = 0$ if $a \in U\pi^{2n}$ for any $n \geq 1$. By considering $\sigma(a) = 1/a$ and $\tau(a) = 1 - a$, it easily follows that $[a]_\chi = 0$ whenever $a \in U_1$ or $v(a) \neq 0$.

Finally, let $a \in U \setminus U_1$ and $w \in U_1$. Then

$$0 = [a]_\chi - [aw]_\chi + \langle a\rangle [w]_\chi - \left\langle a^{-1} - 1\right\rangle \left[w \cdot \frac{1-a}{1-aw}\right]_\chi + \langle 1 - a\rangle \left[\frac{1-a}{1-aw}\right]_\chi$$

which gives $[a]_\chi = [aw]_\chi$ since the last three terms all lie in U_1. This proves the Lemma. □

Lemma 6.12. *Let F be a local field whose residue field k has order q with $q \equiv 1 \pmod 4$. Then*

$$\overline{\mathcal{RP}(F)}[\tfrac{1}{2}]_\psi = \overline{\mathcal{RP}(F)}[\tfrac{1}{2}]_{\psi'} = 0.$$

Proof. We treat the case of ψ. Clearly the case of ψ' is identical since it only involves a switch in our choice of uniformizer.

Since $\psi = \mathrm{e}^-_u \mathrm{e}^-_\pi$ and since $-1 \in (F^\times)^2$ we have

$$[a]_\psi = 0 \text{ if } a \equiv u, \pi \pmod{(F^\times)^2}$$

by Lemma 6.3.

Taking the S_3-action into account as in the last lemma, it follows that

$$[a]_\psi = 0 \text{ if } a \in 1 - R\pi^{2n-1} \cup 1 - N\pi^{2n} \cup R_{-1} \text{ for any } n \geq 1.$$

Consider now $a = r\pi^{2n}$ with $r \in R$ and $n \geq 1$. Then we have

$$0 = \left[\frac{\pi}{a}\right]_\psi - [\pi]_\psi + \langle\pi\rangle [a]_\psi - \left[\frac{a-\pi}{1-\pi}\right]_\psi + \langle\pi\rangle \left[\frac{1}{a} \cdot \frac{a-\pi}{1-\pi}\right]_\psi$$

which forces $\langle\pi\rangle [a]_\psi = 0 = [a]_\psi$ since the other four terms all belong to the square class $\langle\pi\rangle$.

Next, we consider $a = n\pi^{2m-1}$ with $n \in N$ and $m \geq 1$. By Lemma 6.9, we can write $n = r + s$ with $r, s \in R$.

Let $x = r\pi^{2m-1}$. Observe that

$$\begin{aligned} w := \quad & \tfrac{1-x}{1-a} = \tfrac{1-r\pi^{2m-1}}{1-n\pi^{2m-1}} = (1 - r\pi^{2m-1})(1 + n\pi^{2m-1} + \cdots) \\ = \quad & 1 + (n - r)\pi^{2m-1} + \cdots \in 1 + R\pi^{2m-1} = 1 - R\pi^{2m-1} \end{aligned}$$

since $n - r = s \in R$.

Thus we have

$$0 = [x]_\psi - [a]_\psi + \langle\pi\rangle\left[\frac{n}{r}\right]_\psi - \langle\pi\rangle\left[\frac{n}{r}\cdot w\right]_\psi + [w]_\psi = -[a]_\psi$$

since $x \in \langle\pi\rangle\, n/r, nw/r \in N \subset \langle u\rangle$ and, as noted above, $w \in 1 - R\pi^{2m-1}$.

It remains to prove that $[a]_\psi = 0$ whenever $a \in R_1$.

We can write $a = t/s$ with $s, t \in R_{-1}$. (Proof: $\{as \mid s \in R_{-1}\} \cap R_{-1} \neq \emptyset$ since $|\bar{R}_{-1}| = 1 + |\bar{R}_1|$.)

Observe that $(1-s)/(1-t) \in R_{-1}$ also: Since $1-s, 1-t \in N$ we have $(1-s)/(1-t) \in R$. Since $a = s/t \in R_1$,

$$1 - \frac{s}{t} = \frac{t-s}{t} \in R$$

and thus $s-t, t-s \in R$. Hence

$$1 - \frac{1-s}{1-t} = \frac{s-t}{1-t} \in N$$

so that $(1-s)/(1-t) \in R_{-1}$.

Applying the same argument to s^{-1}, t^{-1} shows that $(1-s^{-1})/(1-t^{-1}) \in R_{-1}$ also. Thus

$$0 = [s]_\psi - [t]_\psi + [a]_\psi - \langle u\rangle\left[\frac{1-s^{-1}}{1-t^{-1}}\right]_\psi + \langle u\rangle\left[\frac{1-s}{1-t}\right]_\psi = [a]_\psi$$

proving the lemma. □

Lemma 6.13. *Let F be a local field with residue field k of order q and $q \equiv 3 \pmod 4$. Then*

$$\overline{\mathcal{RB}(F)}[\tfrac{1}{2}]_\psi = \overline{\mathcal{RB}(F)}[\tfrac{1}{2}]_{\psi'} = 0.$$

Proof. By Lemma 6.10 it is enough to show that λ_1 induces isomorphisms of $\mathrm{R}_F[\frac{1}{2}]$-modules

$$\overline{\mathcal{RP}(F)}[\tfrac{1}{2}]_\psi \cong M(F)_\psi \text{ and } \overline{\mathcal{RP}(F)}[\tfrac{1}{2}]_{\psi'} \cong M(F)_{\psi'}.$$

As remarked in the proof of Lemma 6.12, it is enough to treat the case of the character ψ.

Observe that $M(F)_\psi$ is the free $\mathbb{Z}[\frac{1}{2}]$-module of rank 1 with generator $\mathrm{e}_\psi = \mathrm{e}^-_{-1}\mathrm{e}^-_\pi$. Thus, if L is any $\mathrm{R}_F[\frac{1}{2}]$-module and if $a \in L$, there is a unique $\mathrm{R}_F[\frac{1}{2}]$-module homomorphism $M(F)_\psi \to L$ sending e_ψ to $\mathrm{e}_\psi(a)$.

Thus we fix $n \in N_{-1}$ and let $\Phi : M(F)_\psi \to \overline{\mathcal{RP}(F)}[\frac{1}{2}]_\psi$ be the unique $\mathrm{R}_F[\frac{1}{2}]$-module homomorphism sending e_ψ to $\frac{1}{4}[n]_\psi$.

By our previous remarks, Φ is well-defined and clearly

$$\bar{\lambda}_1 \circ \phi = \mathrm{Id}_{M(F)_\psi}.$$

So it only remains to show that ϕ is surjective. To do this we show that $[a]_\psi = 0$ in $\overline{\mathcal{RP}(F)}[\frac{1}{2}]_\psi$ whenever $a \notin N_{-1}$ and that $[n_1]_\psi = [n_2]_\psi$ for any $n_1, n_2 \in N_{-1}$.

Now

$$[a]_\psi = 0 \text{ if } a \equiv 1, -\pi \pmod{(F^\times)^2}.$$

From the action of S_3, it follows easily that

$$[a]_\psi = 0 \text{ for any } a \notin N_{-1}.$$

Finally, let $n_1, n_2 \in N_{-1}$. Then in $\overline{\mathcal{RP}(F)}[\frac{1}{2}]_\psi$ we have

$$0 = [n_1]_\psi - [n_2]_\psi + \langle -1\rangle\left[\frac{n_2}{n_1}\right]_\psi - \langle -1\rangle\left[\frac{1-n_1{}^{-1}}{1-n_2{}^{-1}}\right]_\psi + \left[\frac{1-n_1}{1-n_2}\right]_\psi = [n_1]_\psi - [n_2]_\psi$$

since clearly, from the definition of N_{-1},

$$\frac{n_2}{n_1},\quad \frac{1-n_1}{1-n_2},\quad \frac{1-{n_1}^{-1}}{1-{n_2}^{-1}}\in R.$$

This proves the lemma. □

Putting all of these results together gives the following calculation of the third homology of $\mathrm{SL}_2(F)$:

Theorem 6.14. *Let F be a local field with residue field k of order $q = p^f$, where q is odd. If $\mathbb{Q}_3 \subset F$, suppose that $v_F(3)$ and f are both odd.*

Then there is an isomorphism of $\mathrm{R}_F[\frac{1}{2}]$-modules

$$\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}])\cong K_3^{\mathrm{ind}}(F)[\tfrac{1}{2}]\oplus\mathcal{B}(k)[\tfrac{1}{2}]$$

in which $F^\times$ acts trivially on the first factor, while (the square class) of any uniformizer acts as multiplication by -1 on the second factor.

Proof. As observed above, there is a $\mathrm{R}_F[\frac{1}{2}]$-module decomposition

$$\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}])\cong K_3^{\mathrm{ind}}(F)[\tfrac{1}{2}]\oplus\mathcal{RB}(F)[\tfrac{1}{2}]_\chi\oplus\mathcal{RB}(F)[\tfrac{1}{2}]_\psi\oplus\mathcal{RB}(F)[\tfrac{1}{2}]_{\psi'}.$$

Now

$$\mathcal{RB}(F)[\tfrac{1}{2}]_\chi\cong\widehat{\mathcal{P}(k)}[\tfrac{1}{2}]$$

by Lemma 6.6, Lemma 6.10 and Lemma 6.11.

Furthermore, for a finite field k the kernel and cokernel of the natural map $\mathcal{B}(k)\to\widehat{\mathcal{P}(k)}$ are annihilated by 2 so that there is an induced isomorphism

$$\mathcal{B}(k)[\tfrac{1}{2}]\cong\widehat{\mathcal{P}(k)}[\tfrac{1}{2}].$$

Finally, we have

$$\mathcal{RB}(F)[\tfrac{1}{2}]_\psi=\mathcal{RB}(F)[\tfrac{1}{2}]_{\psi'}=0$$

by Corollary 6.5, Lemma 6.12 and Lemma 6.13.

□

Remark 6.15. If $\mathbb{Q}_3 \subset F$ and at least one if $v_F(3)$ and f is even, then there is a surjective homomorphism of $\mathrm{R}_F[\frac{1}{2}]$-modules

$$\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\tfrac{1}{2}])\twoheadrightarrow K_3^{\mathrm{ind}}(F)[\tfrac{1}{2}]\oplus\mathcal{B}(k)[\tfrac{1}{2}]$$

whose kernel has order 1 or 3.

Remark 6.16. It is not difficult to write down an explicit homology class which generates $\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\frac{1}{2}])_0\cong\mathcal{B}(\mathbb{F}_q)[\frac{1}{2}]$ for a local field F with residue field $\mathbb{F}_q$ of odd order.

Let $r=(q+1)'$. Let $a\in U\setminus U^2$ and let $\alpha+\beta\sqrt{a}\in F(\sqrt{a})$ be a primitive r-th root of unity. Let

$$t=\begin{bmatrix}\alpha & a\beta\\ \beta & \alpha\end{bmatrix}\in\mathrm{SL}_2(F)$$

be the corresponding matrix of order r. Then

$$z=\sum_{i=0}^{n-1}1\otimes(1,t,t^i,t^{i+1})$$

represents a homology class, α, of order r in $\mathrm{H}_3(\mathrm{SL}_2(F),\mathbb{Z}[\frac{1}{2}])$.

Let $\bar{t} \in \mathrm{SL}_2(\mathbb{F}_q)$ be the matrix obtained by reducing the entries of t. Then the class α maps to a generator of $\mathcal{B}(\mathbb{F}_q)[\frac{1}{2}]$ under the specialization homomorphism $S_{\pi,-1}$ since the diagram

$$\begin{tikzcd} \mathrm{H}_3(\langle t\rangle, \mathbb{Z}[\frac{1}{2}]) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) \arrow[r] & \mathcal{RB}(F)[\frac{1}{2}] \arrow[d, "S_{\pi,-1}"] \\ \mathrm{H}_3(\langle \bar{t}\rangle, \mathbb{Z}[\frac{1}{2}]) \arrow[r] & \mathrm{H}_3(\mathrm{SL}_2(\mathbb{F}_q), \mathbb{Z}[\frac{1}{2}]) \arrow[r] & \mathcal{B}(\mathbb{F}_q)[\frac{1}{2}] \end{tikzcd}$$

since the composite map on the bottom row is an isomorphism (see Lemma 2.6 above).

Let π be a uniformizing parameter and let

$$t_\pi = \begin{bmatrix} \pi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & a\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} 1/\pi & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & \pi a\beta \\ \beta/\pi & \alpha \end{bmatrix}.$$

Then the homology class

$$\mathrm{e}^-_\pi(\alpha) = \frac{1}{2}\sum_{i=0}^{n-1} 1 \otimes \Big((1, t, t^i, t^{i+1}) - (1, t_\pi, t_\pi^i, t_\pi^{i+1})\Big)$$

is a generator of $\mathrm{H}_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])_0$ by the arguments of this section.

Remark 6.17. The same techniques as used above apply also to case where the residue field has characteristic 2. However, the number of square classes is of the form 2^k, $k \geq 3$ and the 'S_3-dynamics' become considerably more complicated as k grows. The author has confirmed, for example, that

$$\mathrm{H}_3(\mathrm{SL}_2(\mathbb{Q}_2), \mathbb{Z}[\tfrac{1}{2}]) \cong K_3^{\mathrm{ind}}(\mathbb{Q}_2)[\tfrac{1}{2}] \oplus \mathcal{B}(\mathbb{F}_2)$$

as expected, but the calculations are long and unedifying.

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